

# From simplicial Chern-Simons theory to the shadow invariant II

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## Abstract

This is the second of a series of papers in which we develop a “discretization approach” for the rigorous realization of the non-Abelian Chern-Simons path integral for manifolds  $M$  of the form  $M = \Sigma \times S^1$  and arbitrary simply-connected compact structure groups  $G$ . More precisely, we introduce, for general links  $L$  in  $M$ , a rigorous version  $\text{WLO}_{\text{rig}}(L)$  of (the expectation values of) the corresponding Wilson loop observable  $\text{WLO}(L)$  in the so-called “torus gauge” by Blau and Thompson (Nucl. Phys. B408(1):345–390, 1993). For a simple class of links  $L$  we then evaluate  $\text{WLO}_{\text{rig}}(L)$  explicitly in a non-perturbative way, finding agreement with Turaev’s shadow invariant  $|L|$ .

## 1 Introduction

Recall from [30] that our goal is to find, for manifolds  $M$  of the form  $M = \Sigma \times S^1$ , a rigorous realization of the non-Abelian Chern-Simons path integral in the torus gauge. We want to achieve this with the help of a suitable “discretization approach”. In [30] we introduced already one such approach, “Approach I”. In the present paper we will introduce a second approach, “Approach II”.

The paper is organized as follows: In Sec. 2 we will recall some of the notation from [30] and we will restate the basic heuristic formula from [30], cf. Eq. (2.8) below. Moreover, we will recall some of the constructions in Approach I. In Sec. 3 we will describe and motivate the main changes in Approach II compared to Approach I. In Sec. 4 we will then give the remaining details of Approach II. In Sec. 5 we state and prove our main result, Theorem 5.3, which is concerned with a special class of links. In Sec. 6 we make some remarks regarding the case of general links and in Sec. 7 we then conclude the main part of this paper with an outlook on some promising further directions within the framework of  $BF_3$ -theory.

The present paper has an appendix consisting of six parts: part A contains a list of the Lie theoretic notation which will be relevant in the present paper (this list is a continuation of the one in Appendix A in [30]). In part B we recall the definition of Turaev’s shadow invariant  $|L|$  for links  $L$  in 3-manifolds  $M$  of the type  $M = \Sigma \times S^1$ . In part C we recall the definition of  $BF$ -theory in 3 dimensions and we briefly comment on the relationship between  $BF_3$ -theory and CS theory. In part D we sketch two possible modifications of Approach II. In part E we sketch two alternative versions of Theorem 7.4 of Sec. 7. Finally, in part F we make some general remarks on the so-called “simplicial program” for CS/ $BF_3$ -theory.

## 2 A short review of (the relevant parts of) [30]

### 2.1 Some notation from Sec. 2 in [30]

As in [30] we fix a simply-connected compact Lie group  $G$  and a maximal torus  $T$  of  $G$ . By  $\mathfrak{g}$  and  $\mathfrak{t}$  we will denote the Lie algebras of  $G$  and  $T$  and by  $\langle \cdot, \cdot \rangle$  the unique Ad-invariant scalar product on  $\mathfrak{g}$  satisfying the normalization condition  $\langle \check{\alpha}, \check{\alpha} \rangle = 2$  for every short coroot  $\check{\alpha}$  w.r.t.  $(\mathfrak{g}, \mathfrak{t})$ , cf. part A of the Appendix. For later use let us also fix a Weyl chamber  $\mathcal{C} \subset \mathfrak{t}$ .

Moreover, we will fix a compact oriented 3-manifold  $M$  of the form  $M = \Sigma \times S^1$  where  $\Sigma$  is a (compact oriented) surface, and an ordered oriented link  $L = (l_1, \dots, l_m)$ ,  $m \in \mathbb{N}$ , in  $M = \Sigma \times S^1$ . Each  $l_i$  is “colored” with a finite-dimensional representation  $\rho_i$  of  $G$ .

As in [30] we will use the following notation<sup>1</sup>

$$\mathcal{B} = C^\infty(\Sigma, \mathfrak{t}) = \Omega^0(\Sigma, \mathfrak{t}) \quad (2.1a)$$

$$\mathcal{A} = \Omega^1(M, \mathfrak{g}) \quad (2.1b)$$

$$\mathcal{A}_\Sigma = \Omega^1(\Sigma, \mathfrak{g}) \quad (2.1c)$$

$$\mathcal{A}_{\Sigma, \mathfrak{t}} = \Omega^1(\Sigma, \mathfrak{t}), \quad \mathcal{A}_{\Sigma, \mathfrak{k}} = \Omega^1(\Sigma, \mathfrak{k}) \quad (2.1d)$$

$$\mathcal{A}^\perp = \{A \in \mathcal{A} \mid A(\partial/\partial t) = 0\} \quad (2.1e)$$

$$\check{\mathcal{A}}^\perp = \{A^\perp \in \mathcal{A}^\perp \mid \int A^\perp(t) dt \in \mathcal{A}_{\Sigma, \mathfrak{k}}\} \quad (2.1f)$$

$$\mathcal{A}_c^\perp = \{A^\perp \in \mathcal{A}^\perp \mid A^\perp \text{ is constant and } \mathcal{A}_{\Sigma, \mathfrak{t}}\text{-valued}\} \quad (2.1g)$$

Here  $\mathfrak{k}$  is the orthogonal complement of  $\mathfrak{t}$  in  $\mathfrak{g}$  w.r.t.  $\langle \cdot, \cdot \rangle$ ,  $dt$  is the normalized (translation-invariant) volume form<sup>2</sup> on  $S^1$ ,  $\partial/\partial t$  is the vector field on  $M = \Sigma \times S^1$  obtained by “lifting” the standard vector field  $\partial/\partial t$  on  $S^1$  and in Eqs. (2.1f) and (2.1g) we used the “obvious” identification (cf. Sec. 2.3.1 in [30])

$$\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_\Sigma) \quad (2.2)$$

where  $C^\infty(S^1, \mathcal{A}_\Sigma)$  is the space of maps  $f : S^1 \rightarrow \mathcal{A}_\Sigma$  which are “smooth” in the sense that  $\Sigma \times S^1 \ni (\sigma, t) \mapsto (f(t))(X_\sigma) \in \mathfrak{g}$  is smooth for every smooth vector field  $X$  on  $\Sigma$ . It follows from the definitions above that

$$\mathcal{A}^\perp = \check{\mathcal{A}}^\perp \oplus \mathcal{A}_c^\perp \quad (2.3)$$

Recall that the Chern-Simons action function  $S_{CS} : \mathcal{A} \rightarrow \mathbb{R}$  associated to  $M$ ,  $G$ , and the “level”  $k \in \mathbb{Z} \setminus \{0\}$  is given by<sup>3</sup>

$$S_{CS}(A) = -k\pi \int_M \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle, \quad A \in \mathcal{A} \quad (2.4)$$

where  $[\cdot \wedge \cdot]$  denotes the wedge product associated to the Lie bracket  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  and where  $\langle \cdot \wedge \cdot \rangle$  denotes the wedge product associated to the scalar product  $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ .

By  $\text{Hol}_l(A)$  we will denote the holonomy of  $A \in \mathcal{A}$  around a loop  $l$ . The following explicit formula for  $\text{Hol}_l(A)$  proved to be useful in [30]:

$$\text{Hol}_l(A) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \exp\left(\frac{1}{n} A(l'(t))\right)_{|t=k/n} \quad (2.5)$$

where  $\exp : \mathfrak{g} \rightarrow G$  is the exponential function of  $G$ .

<sup>1</sup>recall that  $\Omega^p(N, V)$  denotes the space of  $V$ -valued  $p$ -forms on a smooth manifold  $N$

<sup>2</sup>or, equivalently, the normalized Haar measure

<sup>3</sup>recall that if  $G$  is not simple then we can generalize the definition of  $S_{CS}$ , Remark 2.7 in [30]

As in Sec. 2.2.4 in [30] we fix a point  $\sigma_0 \in \Sigma$  and associate<sup>4</sup> to each  $h \in [\Sigma, G/T]$ , i.e. each homotopy class<sup>5</sup> of maps  $\Sigma \rightarrow G/T$ , a  $\mathfrak{t}$ -valued 1-form  $A_{\text{sg}}(h)$  on  $\Sigma \setminus \{\sigma_0\}$ .

Finally, recall the following short notation from [30]:

$$S_{CS}(A^\perp, B) := S_{CS}(A^\perp + Bdt) \quad (2.6)$$

$$\text{Hol}_l(A^\perp, B; h) := \text{Hol}_l(A^\perp + A_{\text{sg}}(h) + Bdt) \quad (2.7)$$

for  $B \in \mathcal{B}$ ,  $A^\perp \in \mathcal{A}^\perp$ ,  $h \in [\Sigma, G/T]$  where  $dt$  is the real-valued 1-form on  $M = \Sigma \times S^1$  obtained by pulling back the 1-form  $dt$  on  $S^1$  in the obvious way.

**Remark 2.1** One can assume without loss of generality that  $G$  is a closed subgroup of  $U(\mathbf{N})$  for some  $\mathbf{N} \in \mathbb{N}$ . In the special case where  $G$  is simple we can then rewrite Eq. (2.4) as

$$S_{CS}(A) = k\pi \int_M \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$$

with  $\text{Tr} := c \cdot \text{Tr}_{\text{Mat}(\mathbf{N}, \mathbb{C})}$  where  $c \in \mathbb{R}$  is chosen such that  $\langle A, B \rangle = -\text{Tr}(A \cdot B)$  for all  $A, B \in \mathfrak{g} \subset \mathfrak{u}(N) \subset \text{Mat}(\mathbf{N}, \mathbb{C})$ . Clearly, making this assumption is a bit inelegant but it has some practical advantages, which is why we made use of it in [30]. In the present paper we will use this assumption only at a later stage, namely in part C of the Appendix below (with  $G$  replaced by  $\tilde{G}$ ).

## 2.2 The basic heuristic formula in [30]

Let us make the identification  $\mathcal{A}_{\Sigma, \mathfrak{t}} \cong \mathcal{A}_c^\perp \subset \mathcal{A}^\perp$ . Then we can rewrite the heuristic formula Eq. (2.53) in [30] as

$$\begin{aligned} \text{WLO}(L) \sim & \sum_{h \in [\Sigma, G/T]} \int_{\mathcal{A}_{\Sigma, \mathfrak{t}} \times \mathcal{B}} \left\{ 1_{C^\infty(\Sigma, \mathfrak{t}_{\text{reg}})}(B) \text{Det}_{FP}(B) \right. \\ & \times \left[ \int_{\check{\mathcal{A}}^\perp} \prod_i \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(\check{A}^\perp + A_c, B; h)) \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp \right] \\ & \times \exp(-2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(h), B \rangle) \left. \right\} \exp(iS_{CS}(A_c, B)) (DA_c \otimes DB) \quad (2.8) \end{aligned}$$

where  $\text{Det}_{FP}(B)$  is the informal expression given by

$$\text{Det}_{FP}(B) := \det(1_{\mathfrak{t}} - \exp(\text{ad}(B)))_{|\mathfrak{t}} \quad (2.9)$$

and where  $\langle dA_{\text{sg}}(h), B \rangle := \langle dA_{\text{sg}}(h) \wedge B \rangle$  denotes the wedge product<sup>6</sup> of the  $\mathfrak{t}$ -valued 2-form  $dA_{\text{sg}}(h)$  and the 0-form  $B|_{\Sigma \setminus \{\sigma_0\}}$  on  $\Sigma \setminus \{\sigma_0\}$ .

If we fix an auxiliary Riemannian metric  $\mathbf{g}$  on  $\Sigma$  we obtain scalar products  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}}$  and  $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp}$  on  $\mathcal{A}_{\Sigma, \mathfrak{t}}$  and  $\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_\Sigma)$  in a natural way. Moreover, we then have a well-defined Hodge star operator  $\star : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Sigma$  which induces an operator  $\star : C^\infty(S^1, \mathcal{A}_\Sigma) \rightarrow C^\infty(S^1, \mathcal{A}_\Sigma)$  in the obvious way. According to Eq. (2.47) in [30] we have the following explicit formulas

$$S_{CS}(\check{A}^\perp, B) = \pi k \ll \check{A}^\perp, \star(\frac{\partial}{\partial t} + \text{ad}(B))\check{A}^\perp \gg_{\mathcal{A}^\perp} \quad (2.10)$$

$$S_{CS}(A_c, B) = -2\pi k \ll A_c, \star dB \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}} \quad (2.11)$$

<sup>4</sup>more precisely, we pick a smooth representative  $\bar{g}_h \in C^\infty(\Sigma, G/T)$  of  $h \in [\Sigma, G/T]$  and then set  $A_{\text{sg}}(h) := \pi_{\mathfrak{t}}(\Omega_h^{-1} d\Omega_h)$  where  $\Omega_h : \Sigma \setminus \{\sigma_0\} \rightarrow G$  is an arbitrary (but fixed) smooth lift of  $(\bar{g}_h)|_{\Sigma \setminus \{\sigma_0\}}$

<sup>5</sup>cf. Footnote 13 in [30]

<sup>6</sup>associated to the restricted scalar product  $\langle \cdot, \cdot \rangle : \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$

**Remark 2.2** According to Remark 2.6 in [30] it may be possible to justify the following simplified version of Eq. (2.8) above:

$$\begin{aligned} \text{WLO}(L) \sim & \sum_{x \in I} \int_{\mathcal{A}_{\Sigma, \mathfrak{t}} \times \mathcal{B}} \left\{ 1_{C^\infty(\Sigma, \mathfrak{t}_{\text{reg}})}(B) \text{Det}_{FP}(B) \right. \\ & \times \left[ \int_{\check{\mathcal{A}}^\perp} \prod_i \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(\check{A}^\perp + A_c + Bdt)) \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp \right] \\ & \left. \times \exp(-2\pi i k \langle x, B(\sigma_0) \rangle) \right\} \exp(iS_{CS}(A_c, B))(DA_c \otimes DB) \quad (2.12) \end{aligned}$$

where  $I = \ker(\exp|_{\mathfrak{t}}) \cong \mathbb{Z}^r$ ,  $r := \dim(\mathfrak{t})$ , cf. Eq. (5.43) below.

Let us mention that (an analogue of) the theorem in Sec. 5.11 in [30] can also be derived within a suitable (and straightforward) modification of Approach I in [30] which is based on Eq. (2.12) instead of Eq. (2.8). Similarly, (an analogue of) Theorem 5.3 below can be derived within a suitable modification of Approach II which is based on the obvious<sup>7</sup> simplified version of Eq. (3.6) below.

### 2.3 Some of the constructions introduced in Approach I

Recall that in Sec. 5.1 in [30] we fixed two finite polyhedral cell decompositions  $\mathcal{C}$  and  $\mathcal{C}'$  of  $\Sigma$  which are dual to each other. By  $\mathcal{K}$  and  $\mathcal{K}'$  we denoted the corresponding (oriented) polyhedral cell complexes, i.e.  $\mathcal{K} := (\Sigma, \mathcal{C})$  and  $\mathcal{K}' := (\Sigma, \mathcal{C}')$ .

Moreover, we fixed  $N \in \mathbb{N}$  and used the finite cyclic group  $\mathbb{Z}_N$  with the “obvious”<sup>8</sup> (oriented) graph structure as a discrete analogue of the Lie group  $S^1$ .

Instead of  $\mathcal{K}$  (resp.  $\mathcal{K}'$ ) we usually wrote  $K_1$  (resp.  $K_2$ ) and we set  $K := (K_1, K_2)$ . By  $\mathfrak{F}_p(K_j)$ ,  $0 \leq p \leq 2$  we denoted the set of  $p$ -faces of  $K_j$ , and – for every fixed real vector space  $V$  – we denoted by  $C^p(K_j, V)$  the space of maps  $\mathfrak{F}_p(K_j) \rightarrow V$  (“ $V$ -valued  $p$ -cochains of  $K_j$ ”).

i) In Sec. 5 in [30] we introduced the following spaces

$$\mathcal{B}(K) := C^0(K_1, \mathfrak{t}) \oplus C^0(K_2, \mathfrak{t}) \quad (2.13a)$$

$$\mathcal{A}_\Sigma(K) := C^1(K_1, \mathfrak{g}) \oplus C^1(K_2, \mathfrak{g}) \quad (2.13b)$$

$$\mathcal{A}_{\Sigma, V}(K) := C^1(K_1, V) \oplus C^1(K_2, V) \quad (2.13c)$$

$$\mathcal{A}^\perp(K) = \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(K)) \cong \text{Map}(\mathbb{Z}_N, C^1(K_1, \mathfrak{g}) \oplus C^1(K_2, \mathfrak{g})) \quad (2.13d)$$

$$\check{\mathcal{A}}^\perp(K) := \{A^\perp \in \mathcal{A}^\perp(K) \mid \sum_{t \in \mathbb{Z}_N} A^\perp(t) \in \mathcal{A}_{\Sigma, \mathfrak{t}}(K)\} \quad (2.13e)$$

$$\mathcal{A}_c^\perp(K) := \{A^\perp \in \mathcal{A}^\perp(K) \mid A^\perp(\cdot) \text{ is constant and } \mathcal{A}_{\Sigma, \mathfrak{t}}(K)\text{-valued}\} \quad (2.13f)$$

Observe that we have  $\mathcal{A}_c^\perp(K) \cong \mathcal{A}_{\Sigma, \mathfrak{t}}(K)$  and

$$\mathcal{A}^\perp(K) = \check{\mathcal{A}}^\perp(K) \oplus \mathcal{A}_c^\perp(K) \quad (2.14)$$

Clearly, Eq. (2.14) is a discrete analogue of the decomposition  $\mathcal{A}^\perp = \check{\mathcal{A}}^\perp \oplus \mathcal{A}_c^\perp$ , cf. Eq. (2.3) above.

**Remark 2.3** Recall from Remark 5.1 in [30] that instead of the space  $\mathcal{A}^\perp(K)$  above we could also work with the space  $\mathcal{A}_{\text{altern}}^\perp(K) := \text{Map}(\mathbb{Z}_{2N}^{\text{odd}}, C^1(K_1, \mathfrak{g})) \oplus \text{Map}(\mathbb{Z}_{2N}^{\text{even}}, C^1(K_2, \mathfrak{g}))$ , cf. Appendix D in [30].  $\mathcal{A}^\perp(K)$  arises from  $\mathcal{A}_{\text{altern}}^\perp(K)$  after making the obvious identifications and has the advantage of simplifying the notation.

<sup>7</sup> i.e. the analogue of Eq. (3.6) which one obtains by rewriting Eq. (2.12) in a similar way as Eq. (3.6) was obtained by rewriting Eq. (2.8)

<sup>8</sup>i.e. the set of edges is given by  $\{(t, t+1) \mid t \in \mathbb{Z}_N\}$

We denote by  $(A_c)_j \in C^1(K_j, V)$  and  $A_j^\perp \in \text{Map}(\mathbb{Z}_N, C^1(K_j, \mathfrak{g}))$ ,  $j \in \{1, 2\}$ , the obvious components of  $A_c \in \mathcal{A}_{\Sigma, V}(K)$  and  $A^\perp \in \mathcal{A}^\perp(K)$ . Moreover, for  $A_c \in \mathcal{A}_{\Sigma, V}(K)$  and  $e \in \mathfrak{F}_1(K_1) \cup \mathfrak{F}_1(K_2)$  we set

$$A_c(e) := (A_c)_j(e) \quad \text{where } j \in \{1, 2\} \text{ is given by } e \in \mathfrak{F}_1(K_j) \quad (2.15)$$

ii) The scalar product  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$  on  $\mathfrak{g}$  induces a scalar product  $\ll \cdot, \cdot \gg_{\mathcal{A}_\Sigma(K)}$  on  $\mathcal{A}_\Sigma(K)$  in the obvious way. Moreover, we obtain a scalar product  $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp(K)}$  on  $\mathcal{A}^\perp(K)$  by

$$\ll A_1^\perp, A_2^\perp \gg_{\mathcal{A}^\perp(K)} := \frac{1}{2N} \sum_{t \in \mathbb{Z}_N} \ll A_1^\perp(t), A_2^\perp(t) \gg_{\mathcal{A}_\Sigma(K)} \quad (2.16)$$

(cf. Remark 5.1 and Appendix D in [30] for the motivation of the factor  $\frac{1}{2N}$  appearing above).

The restriction of  $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp(K)}$  onto  $\check{\mathcal{A}}^\perp(K)$  will be denoted by  $\ll \cdot, \cdot \gg_{\check{\mathcal{A}}^\perp(K)}$ .

iii) Recall also that for every real vector space  $V$  and  $p = 0, 1, 2$  we introduced discrete Hodge operators  $\star_{K_1} : C^p(K_1, V) \rightarrow C^{2-p}(K_2, V)$  and  $\star_{K_2} : C^p(K_2, V) \rightarrow C^{2-p}(K_1, V)$ , cf. Sec. 4 in [30]. Moreover, we introduced two different operators denoted by  $\star_K$ . Firstly, the operator  $\star_K : \mathcal{A}_\Sigma(K) \rightarrow \mathcal{A}_\Sigma(K) = C^1(K_1, \mathfrak{g}) \oplus C^1(K_2, \mathfrak{g})$  given by

$$\star_K := \begin{pmatrix} 0 & \star_{K_2} \\ \star_{K_1} & 0 \end{pmatrix} \quad (2.17)$$

and, secondly, the operator  $\star_K : \mathcal{A}^\perp(K) \rightarrow \mathcal{A}^\perp(K)$  given by

$$(\star_K A^\perp)(t) = \star_K(A^\perp(t)) \quad \forall A^\perp \in \mathcal{A}^\perp(K), t \in \mathbb{Z}_N$$

**Remark 2.4** In view of the discussion in Sec. 3.4 below where we compare Approach II, which will be introduced below, with the “old” Approach I of [30] let us remark that the expression  $\ll A^\perp, \star_K d_K B \gg_{\mathcal{A}^\perp(K)}$  appearing in Eq. (5.8) in [30] could have been rewritten as  $\ll \star_K d_K A^\perp, B \gg_{\mathcal{B}(K)}$  where  $\ll \cdot, \cdot \gg_{\mathcal{B}(K)}$  is the “obvious” scalar product on  $\mathcal{B}(K)$  and where

$$\star_K : C^2(K_1, \mathfrak{t}) \oplus C^2(K_2, \mathfrak{t}) \rightarrow C^0(K_1, \mathfrak{t}) \oplus C^0(K_2, \mathfrak{t}) \quad (2.18)$$

is again given by Eq. (2.17) (with the obvious reinterpretation of the maps  $\star_{K_j}$ ).

### 3 Approach II: Motivation and Overview

The main changes of Approach II compared to Approach I are:

- (CH1) We will not only work with the cell complexes  $K_1 = \mathcal{K}$  and  $K_2 = \mathcal{K}'$  introduced above but also with the barycentric subdivision<sup>9</sup>  $b\mathcal{K}$  of  $\mathcal{K}$ .
- (CH2) We take as our starting point for the rigorous definition of  $\text{WLO}(L)$  a new heuristic formula for  $\text{WLO}(L)$ , which is equivalent to Eq. (2.8) above but suggests a different way of discretizing the corresponding RHS.

These changes will allow us to

- eliminate<sup>10</sup> Modifications (Mod2) and (Mod3) in Approach I
- replace Modification (Mod1) in Approach I by a more natural version,

cf. Remark 5.6 in [30] and Remark 3.11 and Remark 4.1 below.

<sup>9</sup>instead of  $b\mathcal{K}$  we could also work with the coarser cell complex  $q\mathcal{K}$  appearing in Remark 3.1 below

<sup>10</sup>cf., however, Remark 3.11 below

### 3.1 Change (CH1)

From now on we will assume for simplicity that  $K_2 = (\Sigma, \mathcal{C}')$  was chosen to be the “canonical” dual of  $K_1 = (\Sigma, \mathcal{C})$ . In this case the barycentric subdivision of  $\mathcal{C}$  coincides with the one of  $\mathcal{C}'$ . We will denote it by  $b\mathcal{C}$  and we will set  $b\mathcal{K} = (\Sigma, b\mathcal{C})$  in the following. Observe that we have

$$\mathfrak{F}_0(b\mathcal{K}) = \mathfrak{F}_0(K_1) \sqcup \mathfrak{F}_0(K_1|K_2) \sqcup \mathfrak{F}_0(K_2) \quad (3.1)$$

with

$$\mathfrak{F}_0(K_1|K_2) := \{\bar{e} \mid e \in \mathfrak{F}_1(K_1)\} = \{\bar{e} \mid e \in \mathfrak{F}_1(K_2)\} \quad (3.2)$$

where  $\sqcup$  denotes disjoint union and  $\bar{e}$  the barycenter of the edge  $e$ .

Recall that in Approach I we used the space  $\mathcal{B}(K)$  as a discrete analogue for the space  $\mathcal{B}$  appearing in Eq. (2.1). In Approach II we will work with the space

$$\mathcal{B}(b\mathcal{K}) := C^0(b\mathcal{K}, \mathfrak{t}) \quad (3.3)$$

instead. Similarly, we will take as the discrete analogues of the spaces  $\mathcal{A}_{\Sigma, \mathfrak{t}}$  and  $\mathcal{E} := \Omega^2(\Sigma, \mathfrak{t})$  appearing in Eq. (2.8) above and Eq. (3.6) below the spaces

$$\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) := C^1(b\mathcal{K}, \mathfrak{t}) \quad (3.4)$$

$$\mathcal{E}(b\mathcal{K}) := C^2(b\mathcal{K}, \mathfrak{t}) \quad (3.5)$$

The scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}(\supset \mathfrak{t})$  induces scalar products  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})}$  and  $\ll \cdot, \cdot \gg_{\mathcal{E}(b\mathcal{K})}$  on  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  and  $\mathcal{E}(b\mathcal{K})$  in the obvious way.

Recall from [30] that  $C_p(K_1), C_p(K_2), C_p(b\mathcal{K})$   $p \in \{0, 1, 2\}$  denote the corresponding spaces<sup>11</sup> of  $p$ -chains with coefficients in  $\mathbb{R}$ .

**Convention 1** *i) In the following we will identify  $C_p(K_j)$  for  $p \in \{0, 1, 2\}$  and  $j \in \{1, 2\}$  with the “obvious”<sup>12</sup> subspace of  $C_p(b\mathcal{K})$ .*

*ii) For  $p \in \{0, 1, 2\}$  we set*

$$C_p(K) := C_p(K_1) + C_p(K_2) \subset C_p(b\mathcal{K})$$

*Observe that for  $p \in \{0, 1\}$  the sum on the RHS is direct (but for  $p = 2$  it is not). So for  $p \in \{0, 1\}$  we have  $C_p(K) \cong C_p(K_1) \oplus C_p(K_2)$*

*iii) We will identify  $\mathcal{B}(K) = C^0(K_1, \mathfrak{t}) \oplus C^0(K_2, \mathfrak{t}) \cong C_0(K) \otimes_{\mathbb{R}} \mathfrak{t}$  with the obvious subspace of  $\mathcal{B}(b\mathcal{K}) \cong C_0(b\mathcal{K}) \otimes_{\mathbb{R}} \mathfrak{t}$ .*

**Remark 3.1** Instead of working with  $b\mathcal{K}$  we could also work with the “coarser” polyhedral cell complex<sup>13</sup>  $q\mathcal{K}$  on  $\Sigma$  which is determined by

- $\mathfrak{F}_0(q\mathcal{K}) = \mathfrak{F}_0(b\mathcal{K})$
- $\mathfrak{F}_1(q\mathcal{K}) = \mathfrak{F}_1(b\mathcal{K}) \setminus \{e \in \mathfrak{F}_1(b\mathcal{K}) \mid \text{both endpoints of } e \text{ lie in } \mathfrak{F}_0(K_1) \sqcup \mathfrak{F}_0(K_2)\},$

The set  $\mathfrak{F}_2(b\mathcal{K})$  is uniquely determined by  $\mathfrak{F}_0(q\mathcal{K})$  and  $\mathfrak{F}_1(q\mathcal{K})$ . Observe that each  $F \in \mathfrak{F}_2(q\mathcal{K})$  is a quadrangle and that each  $F$  is the union of exactly two faces of  $\mathfrak{F}_2(b\mathcal{K})$ . The use of  $q\mathcal{K}$  would have an important advantage, cf. part D of the Appendix below. However, since the reader is more familiar with  $b\mathcal{K}$  we decided to work with  $b\mathcal{K}$  in the main part of the paper and we will use  $q\mathcal{K}$  only in part D of the Appendix.

<sup>11</sup>since all these cell complexes are finite we can (and sometimes will) identify the spaces above with the spaces  $C^p(K_j, \mathbb{R})$ ,  $j = 1, 2$  and  $C^p(b\mathcal{K}, \mathbb{R})$ , respectively

<sup>12</sup>somewhat more precisely:  $C_p(K_j)$  is identified with the space  $\psi(C_p(K_j))$  where  $\psi : C_p(K_j) \rightarrow C_p(b\mathcal{K})$  is the (uniquely determined) injective linear map  $\psi(\alpha) = \sum_{\alpha'} \alpha' \quad \forall \alpha \in \mathfrak{F}_p(K_j) \subset C_p(K_j)$  where the sum is over those  $\alpha' \in \mathfrak{F}_p(b\mathcal{K})$  which are “contained” in  $\alpha$

<sup>13</sup>cf. Footnote 39 in Sec. 5.2 for an example

## 3.2 Change (CH2)

### 3.2.1 The new heuristic formula for $\text{WLO}(L)$

For reasons which will become clear later (cf. Remark 3.5 in Sec. 3.2.3 below) let us now rewrite Eq. (2.8) using a suitable change of variable. In order to do so we use the fact that the Hodge star operator  $\star : \Omega^2(\Sigma, \mathfrak{t}) \rightarrow \Omega^0(\Sigma, \mathfrak{t})$  (induced by the auxiliary Riemannian metric  $\mathbf{g}$  on  $\Sigma$  fixed in Sec. 2.2 above) is a linear isomorphism. Then we arrive at

$$\begin{aligned} \text{WLO}(L) \sim & \sum_{\mathbf{h} \in [\Sigma, G/T]} \int_{\mathcal{A}_{\Sigma, \mathfrak{t}} \times \mathcal{E}} \left\{ 1_{C^\infty(\Sigma, \text{treg})}(B) \text{Det}_{FP}(B) \right. \\ & \times \left[ \int_{\check{\mathcal{A}}^\perp} \prod_i \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(\check{A}^\perp, A_c, B; \mathbf{h})) \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp \right] \Big|_{B=\star E} \\ & \times \exp(-2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(\mathbf{h}), \star E \rangle \exp(iS_{CS}(A_c, E))(DA_c \otimes DE) \end{aligned} \quad (3.6)$$

where  $DE$  is the informal Lebesgue measure on  $\mathcal{E} := \Omega^2(\Sigma, \mathfrak{t})$  and where we have set

$$S_{CS}(A_c, E) := S_{CS}(A_c, \star E), \quad (3.7)$$

$$\text{Hol}_{l_i}(\check{A}^\perp, A_c, B; \mathbf{h}) := \text{Hol}_{l_i}(\check{A}^\perp + A_c, B; \mathbf{h}) \quad (3.8)$$

From Eq. (2.11) in Sec. 2 above we obtain

$$S_{CS}(A_c, E) = -2\pi k \ll A_c, \delta E \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}} \quad (3.9)$$

where  $\delta : \Omega^2(\Sigma, \mathfrak{t}) \rightarrow \Omega^1(\Sigma, \mathfrak{t}) = \mathcal{A}_{\Sigma, \mathfrak{t}}$  is given by  $\delta := \star d \star$ .

**Remark 3.2** Recall that according to Remark 2.2 above it is probably possible to justify a simplified version of the heuristic equation Eq. (2.8) from which Eq. (3.6) was derived. If so, then we obtain an analogous simplification of Eq. (3.6) above.

### 3.2.2 The map $\partial_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$

Since  $b\mathcal{K}$  is a finite simplicial complex we can make the identifications  $\mathcal{E}(b\mathcal{K}) = C^2(b\mathcal{K}, \mathfrak{t}) \cong C_2(b\mathcal{K}) \otimes \mathfrak{t}$  and  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) = C^1(b\mathcal{K}, \mathfrak{t}) \cong C_1(b\mathcal{K}) \otimes \mathfrak{t}$ . The boundary operator  $\partial_{b\mathcal{K}} : C_2(b\mathcal{K}) \rightarrow C_1(b\mathcal{K})$  then induces in the obvious way an operator  $\mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$ , which will also be denoted by  $\partial_{b\mathcal{K}}$ .

This operator  $\partial_{b\mathcal{K}}$  will be used as the discrete analogue of the operator  $\delta : \Omega^2(\Sigma, \mathfrak{t}) \rightarrow \Omega^1(\Sigma, \mathfrak{t}) = \mathcal{A}_{\Sigma, \mathfrak{t}}$  appearing in Sec. 3.2.1 above.

**Remark 3.3** Let us emphasize that  $\partial_{b\mathcal{K}}$  is indeed a natural discrete analogue of the operator  $\delta$  appearing in Eq. (3.9) above. In order to see this note, firstly, that  $\partial_{b\mathcal{K}}$  is clearly a natural discrete analogue of the differential  $d : \Omega^1(\Sigma, \mathfrak{t}) \rightarrow \Omega^2(\Sigma, \mathfrak{t})$ , secondly, that  $\delta$  is the adjoint of  $d$  w.r.t. to the two scalar products  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}}$  and  $\ll \cdot, \cdot \gg_{\mathcal{E}}$  on  $\mathcal{A}_{\Sigma, \mathfrak{t}}$  and  $\mathcal{E}$  and, thirdly, that  $\partial_{b\mathcal{K}}$  is the adjoint of  $d_{b\mathcal{K}}$  w.r.t. to the two scalar products  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})}$  and  $\ll \cdot, \cdot \gg_{\mathcal{E}(b\mathcal{K})}$  on  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  and  $\mathcal{E}(b\mathcal{K})$ .

Put differently: we can rewrite Eq. (3.9) above as

$$S_{CS}(A_c, E) = -2\pi k \ll dA_c, E \gg_{\mathcal{E}} \quad (3.10)$$

This suggests that for the discrete analogue  $S_{CS}^{disc}(A_c, E)$  of  $S_{CS}(A_c, E)$  we make the ansatz

$$S_{CS}^{disc}(A_c, E) := -2\pi k \ll d_{b\mathcal{K}} A_c, E \gg_{\mathcal{E}(b\mathcal{K})} \quad (3.11)$$

But since  $\partial_{b\mathcal{K}}$  is the adjoint of  $d_{b\mathcal{K}}$  w.r.t. the scalar products  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})}$  and  $\ll \cdot, \cdot \gg_{\mathcal{E}(b\mathcal{K})}$  the RHS of the last equation coincides with the RHS of Eq. (4.2b) in Sec. 4.2 below.

<sup>14</sup>the scalar product  $\ll \cdot, \cdot \gg_{\mathcal{E}}$  is defined in the obvious way



### 3.2.3 The map $\star_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{B}(b\mathcal{K})$

The map  $\star_K : C^2(K_1, \mathfrak{t}) \oplus C^2(K_2, \mathfrak{t}) \rightarrow C^0(K_2, \mathfrak{t}) \oplus C^0(K_1, \mathfrak{t})$  appearing in Remark 2.4 above can be considered as a discrete analogue of the map  $\star : \Omega^2(\Sigma, \mathfrak{t}) \rightarrow \Omega^0(\Sigma, \mathfrak{t})$  appearing in Sec. 3.2.1 above. However, in Approach II this map<sup>15</sup>  $\star_K$  will not be useful. Instead we will work with a suitable map  $\star_{b\mathcal{K}} : C^2(b\mathcal{K}, \mathfrak{t}) \rightarrow C^0(b\mathcal{K}, \mathfrak{t})$ , which we will now introduce. In order to do so we proceed in two steps:

**Step 1** First we observe that the (simplicial) cell decomposition  $b\mathcal{C}$  appearing in  $b\mathcal{K} = (\Sigma, b\mathcal{C})$  induces in a canonical way a “normalized”<sup>16</sup> Riemannian metric on

$$\Sigma^{(2)} := \Sigma^{(2)}(b\mathcal{C}) := \text{union of the open 2-simplices of } b\mathcal{C} \quad (3.12)$$

Let  $\mathbf{g}_{can}$  be this Riemannian metric and let  $\star_{can} : \Omega^2(\Sigma^{(2)}, \mathfrak{t}) \rightarrow \Omega^0(\Sigma^{(2)}, \mathfrak{t})$  be the Hodge star operator w.r.t.  $\mathbf{g}_{can}$ . We also observe that there is a canonical map, the so-called Whitney map,  $W : C^2(b\mathcal{K}, \mathfrak{t}) \rightarrow \Omega^2(\Sigma^{(2)}, \mathfrak{t})$ .

Thus we obtain a natural map  $\star : C^2(b\mathcal{K}, \mathfrak{t}) \rightarrow Map(\Sigma^{(2)}, \mathfrak{t})$  given by  $\star E := \star_{can} W(E)$  for all  $E \in C^2(b\mathcal{K}, \mathfrak{t})$ . In fact, assuming the aforementioned normalization condition on the Riemannian metric this map can be rewritten explicitly by the simple equation

$$(\star E)(x) = E(F_x) \quad (3.13)$$

for all  $x \in \Sigma^{(2)}$  where  $F_x$  is the unique 2-face containing  $x$ .

**Step 2** Recall that, for each  $E \in C^2(b\mathcal{K}, \mathfrak{t})$ , the map  $\star E$  is only defined on  $\Sigma^{(2)}$  and not on all of  $\Sigma$ . In order to get a map<sup>17</sup>  $C^2(b\mathcal{K}, \mathfrak{t}) \rightarrow C^0(b\mathcal{K}, \mathfrak{t})$  we will now extend each  $\star E$  to a map  $\overline{\star E} : \Sigma \rightarrow \mathfrak{t}$ .

In order to do so we make the ansatz

$$\overline{\star E}(x) := \begin{cases} \text{mean}_{\{F \in \mathfrak{F}_2(b\mathcal{K}) | x \in \overline{F}\}} E(F) & \text{if } x \in \mathcal{O} \\ \text{mean}_{\{F \in \mathfrak{F}_2(b\mathcal{K}) | x \in \overline{F}, F \cap \mathcal{O} = \emptyset\}} E(F) & \text{if } x \notin \mathcal{O} \end{cases} \quad (3.14)$$

where  $\mathcal{O} \subset \Sigma$  is a suitable open subset of  $\Sigma$  which will be fixed below, cf. part i) of Remark 3.4 below for a comment on  $\mathcal{O}$ .

The map  $C^2(b\mathcal{K}, \mathfrak{t}) \ni E \mapsto \overline{\star E} \in Map(\Sigma, \mathfrak{t})$  now induces a map  $\mathcal{E}(b\mathcal{K}) = C^2(b\mathcal{K}, \mathfrak{t}) \rightarrow C^0(b\mathcal{K}, \mathfrak{t}) = \mathcal{B}(b\mathcal{K})$  in the obvious way. Both of the two maps mentioned in the last sentence will be denoted by  $\star_{b\mathcal{K}}$  in the following. Moreover, also the two analogous maps where  $\mathfrak{t}$  is replaced by  $\mathbb{R}$  will be denoted by  $\star_{b\mathcal{K}}$  in the following.

**Remark 3.4** i) The set  $\mathcal{O}$  appearing above will be taken to be  $\mathcal{O} = \bigcup_i O_i$  where  $O_i \subset \Sigma$  are as in Sec. 5.2. Accordingly, the definition of  $\star_{b\mathcal{K}}$  will depend on the link  $L = (l_1, l_2, \dots, l_m)$  and the framings  $(l'_1, l'_2, \dots, l'_m)$  fixed in Sec. 4.8 below.

ii) For those<sup>18</sup>  $E \in \mathcal{E}(b\mathcal{K})$  which appear in the proof of Theorem 5.3 below the arithmetic means appearing on the RHS of Eq. (3.14) will be trivial. In this situation  $\overline{\star E}$  can be characterized as the unique extension of  $\star E : \Sigma^{(2)} \rightarrow \mathfrak{t}$  to all of  $\Sigma$  such that both  $\overline{\star E}|_{\Sigma \setminus \mathcal{O}}$  and  $\overline{\star E}|_{\mathcal{O}}$  are continuous.

<sup>15</sup>which should not be confused with the map  $\star_K : \mathcal{A}^\perp(K) \rightarrow \mathcal{A}^\perp(K)$  which appears both in Approach I and Approach II

<sup>16</sup>we choose the normalization such that every 2-simplex has area 1

<sup>17</sup>recall that  $C^0(b\mathcal{K}, \mathfrak{t})$  is the space of maps  $\mathfrak{F}_0(b\mathcal{K}) \rightarrow \mathfrak{t}$  but  $\mathfrak{F}_0(b\mathcal{K}) \cap \Sigma^{(2)} = \emptyset$

<sup>18</sup>i.e.  $E$  of the form  $E = b + \frac{1}{k} \sum_i \alpha_i \cdot D_i$  where  $b \in \mathfrak{t}$ ,  $\alpha_i \in \Lambda$  and  $D_i$  given as in Sec. 5.4.2 below



**Remark 3.5** I have not been able to find a discrete analogue  $C^0(b\mathcal{K}, \mathfrak{t}) \rightarrow C^2(b\mathcal{K}, \mathfrak{t})$  of the map  $\star : \mathcal{B} \rightarrow \mathcal{E}$  which would lead to a result like Theorem 5.3. This is the main reason for making the transition from the space  $\mathcal{B}$  to  $\mathcal{E}$  in the heuristic formula (3.6) above. This transition allows us work with the discrete analogue<sup>19</sup>  $\star_{b\mathcal{K}} : C^2(b\mathcal{K}, \mathfrak{t}) \rightarrow C^0(b\mathcal{K}, \mathfrak{t})$  of the map  $\star : \mathcal{E} \rightarrow \mathcal{B}$  instead.

### 3.3 New discretization of the operator $\frac{\partial}{\partial t} + \text{ad}(B)$ appearing in Eq. (2.10)

Let us now reconsider the issue of discretizing the operator<sup>20</sup>  $\frac{\partial}{\partial t} + \text{ad}(B)$ , appearing in Eq. (2.10) above. We will often write  $\partial_t$  instead of  $\frac{\partial}{\partial t}$  in the following.

As a preparation let us consider first, for fixed  $b \in \mathfrak{t}$ , the continuum operator  $L(b) := \partial_t + \text{ad}(b) : C^\infty(S^1, \mathfrak{g}) \rightarrow C^\infty(S^1, \mathfrak{g})$ . We want to find discrete analogue  $L^N(b) : \text{Map}(\mathbb{Z}_N, \mathfrak{g}) \rightarrow \text{Map}(\mathbb{Z}_N, \mathfrak{g})$ .

**Remark 3.6** Recall that we have identified  $\mathbb{Z}_N$  with the subgroup  $\{e^{\frac{2\pi i}{N}k} \mid 1 \leq k \leq N\}$  of the Lie group<sup>21</sup>  $S^1$ . Note that under this identification  $1 \in \mathbb{Z}_N$  is identified with  $e^{2\pi i \frac{1}{N}} = i_{S^1}(1/N) \in S^1$ .

Three natural choices for  $L^N(b)$  are

$$\hat{\partial}_t^{(N)} + \text{ad}(b), \quad \check{\partial}_t^{(N)} + \text{ad}(b), \quad \bar{\partial}_t^{(N)} + \text{ad}(b)$$

with

$$\hat{\partial}_t^{(N)} := N(\tau_1 - \tau_0), \quad \check{\partial}_t^{(N)} := N(\tau_0 - \tau_{-1}), \quad \bar{\partial}_t^{(N)} := \frac{N}{2}(\tau_1 - \tau_{-1})$$

where  $\tau_x$  is the translation operator  $\text{Map}(\mathbb{Z}_N, \mathfrak{g}) \rightarrow \text{Map}(\mathbb{Z}_N, \mathfrak{g})$  given by  $(\tau_x f)(t) = f(t + x)$ . (Instead of  $\tau_0$  we will simply write 1 in the following).

The discretization of  $\partial_t + \text{ad}(B)$  introduced in Sec. 5.2 in [30] was closely related to the two operators  $\hat{\partial}_t^{(N)} + \text{ad}(b)$  and  $\check{\partial}_t^{(N)} + \text{ad}(b)$ .

In fact, there are other quite natural discrete versions of  $L(b) = \partial_t + \text{ad}(b)$ , namely

$$\hat{L}^{(N)}(b) := N(\tau_1 e^{\text{ad}(b)/N} - 1) \tag{3.15a}$$

$$\check{L}^{(N)}(b) := N(1 - \tau_{-1} e^{-\text{ad}(b)/N}) \tag{3.15b}$$

$$\bar{L}^{(N)}(b) := \frac{N}{2}(\tau_1 e^{\text{ad}(b)/N} - \tau_{-1} e^{-\text{ad}(b)/N}) \quad \text{if } N \text{ is even} \tag{3.15c}$$

and – as we will see later – working with the latter three operator allows us to avoid the continuum limit in the  $S^1$ -direction mentioned in Remark 5.9 in [30].

We will now restrict our attention to the first of these three operators, i.e.  $\hat{L}^{(N)}(b)$ , and we will demonstrate that this operator is indeed a natural discretization of  $L(b)$ . Similar considerations can be made for the other two operators  $\check{L}^{(N)}(b)$  and  $\bar{L}^{(N)}(b)$ .

#### 3.3.1 1. Motivation

Let  $b \in \mathfrak{t}$  be fixed and let  $(T_s)_{s \in \mathbb{R}}$  be the 1-parameter group of orthogonal operators on the real Hilbert space  $L^2_{\mathfrak{g}}(S^1, dt)$  which is generated by  $L(b)$ . We have the following explicit formulas:

$$T_s = \tau_{i(s)} e^{s \text{ad}(b)}, \quad s \in \mathbb{R} \tag{3.16}$$

<sup>19</sup>in contrast to  $K_1$  or  $K_2$  the polyhedral cell complex  $b\mathcal{K}$  will always be a simplicial complex, in particular we will always have  $\#\mathfrak{F}_2(b\mathcal{K}) \geq \#\mathfrak{F}_0(b\mathcal{K})$  where  $=$  holds only in very few special cases. In view of this asymmetry it should not be surprising that, apparently, it makes a difference if one tries to find a “good” discrete analogue of  $\star : \mathcal{B} \rightarrow \mathcal{E}$  or of  $\star : \mathcal{E} \rightarrow \mathcal{B}$

<sup>20</sup> Recall that in [30] we introduced the map  $i_{S^1} : \mathbb{R} \ni s \mapsto e^{2\pi i s} \in S^1$ . We will often simply write  $i(s)$  instead of  $i_{S^1}(s)$ ,  $s \in \mathbb{R}$ . Recall also that  $\frac{\partial}{\partial t}$  is the vector field on  $S^1$  induced by  $i_{S^1}$

<sup>21</sup>We will write the group law of  $S^1$  additively

$$L(b) = \lim_{s \rightarrow 0} \frac{T_s - T_0}{s} \quad \text{on } C^\infty(S^1, \mathfrak{g}) \quad (3.17)$$

where  $\tau_t$  is the translation operator  $L_{\mathfrak{g}}^2(S^1, dt) \rightarrow L_{\mathfrak{g}}^2(S^1, dt)$  given by  $(\tau_t f)(t') = f(t + t')$ .

As a discrete analogue  $(T_s^{(N)})_{s \in \frac{1}{N}\mathbb{Z}}$  of  $(T_s)_{s \in \mathbb{R}}$  we now take

$$T_s^{(N)} = \tau_{i(s)} e^{s \operatorname{ad}(b)}, \quad s \in \frac{1}{N}\mathbb{Z}$$

and a natural discrete analogue of the RHS of Eq. (3.17) is then

$$\frac{T_{1/N}^{(N)} - T_0^{(N)}}{1/N} = N(\tau_{i(1/N)} e^{\frac{1}{N} \operatorname{ad}(b)} - 1) = \hat{L}^{(N)}(b)$$

### 3.3.2 2. Motivation

Let  $b \in \mathfrak{t}$  be fixed. Observe that  $\hat{L}^{(N)}(b)$  coincides with  $\hat{\partial}_t$  on  $\operatorname{Map}(\mathbb{Z}_N, \mathfrak{t})$ , which is a very natural operator. For our purposes it is therefore enough to demonstrate that the operator

$$S^{(N)} := \hat{L}^{(N)}(b)|_{\operatorname{Map}(\mathbb{Z}_N, \mathfrak{t})}$$

is a natural discretization of the continuum operator

$$S := L(b)|_{C^\infty(S^1, \mathfrak{t})}$$

In the special case where  $b \in \mathfrak{t}_{reg}$  (which is the only case relevant for us)  $S$  is invertible and  $S^{-1} : C^\infty(S^1, \mathfrak{t}) \rightarrow C^\infty(S^1, \mathfrak{t})$  is given explicitly by

$$(S^{-1}f)(t) = (\exp(\operatorname{ad}(b))|_{\mathfrak{t}} - 1_{\mathfrak{t}})^{-1} \cdot \int_0^1 e^{s \operatorname{ad}(b)} f(t + i(s)) ds, \quad t \in S^1 \quad (3.18)$$

This suggests the following discrete analogue  $S_{(N)}^{-1} : \operatorname{Map}(\mathbb{Z}_N, \mathfrak{t}) \rightarrow \operatorname{Map}(\mathbb{Z}_N, \mathfrak{t})$  of  $S^{-1}$ :

$$(S_{(N)}^{-1}f)(t) = (\exp(\operatorname{ad}(b))|_{\mathfrak{t}} - 1_{\mathfrak{t}})^{-1} \cdot \frac{1}{N} \sum_{k=0}^{N-1} [e^{s \operatorname{ad}(b)} f(t + i(s))]_{|s=k/N}, \quad t \in \mathbb{Z}_N \quad (3.19)$$

Clearly, if  $S_{(N)}^{-1}$  is invertible then  $(S_{(N)}^{-1})^{-1}$  can be considered as a discrete analogue of  $S$ .

Now a short computation shows that  $S_{(N)}^{-1} \cdot S^{(N)} = \operatorname{id}_{\operatorname{Map}(\mathbb{Z}_N, \mathfrak{t})}$  so  $S_{(N)}^{-1}$  is indeed invertible and we have  $S^{(N)} = (S_{(N)}^{-1})^{-1}$ .

### 3.3.3 Computing the determinants

Let  $b \in \mathfrak{t}$  again be fixed. Recall the definition of the three linear operators  $\hat{L}^{(N)}(b)$ ,  $\check{L}^{(N)}(b)$ , and  $\bar{L}^{(N)}(b)$  on  $\operatorname{Map}(\mathbb{Z}_N, \mathfrak{g})$ , cf. Eqs. (3.15) above.

In the following we will consider the restriction of each of these three operators onto the orthogonal<sup>22</sup> complement of its kernel<sup>23</sup>. The restrictions will again be denoted by  $\hat{L}^{(N)}(b)$ ,  $\check{L}^{(N)}(b)$ , and  $\bar{L}^{(N)}(b)$ .

<sup>22</sup>w.r.t. the obvious scalar product on  $\operatorname{Map}(\mathbb{Z}_N, \mathfrak{g})$

<sup>23</sup>for  $b \in \mathfrak{t}_{reg}$ , which is the case relevant for us, we have  $\ker(\hat{L}^{(N)}(b)) = \ker(\check{L}^{(N)}(b)) = \operatorname{Map}_c(\mathbb{Z}_N, \mathfrak{t})$  with  $\operatorname{Map}_c(\mathbb{Z}_N, \mathfrak{t}) := \{f \in \operatorname{Map}(\mathbb{Z}_N, \mathfrak{g}) \mid f \text{ is a constant function taking values in } \mathfrak{t}\} \cong \mathfrak{t}$ . The orthogonal complement  $\operatorname{Map}'(\mathbb{Z}_N, \mathfrak{g})$  of  $\operatorname{Map}_c(\mathbb{Z}_N, \mathfrak{t})$  is given by  $\operatorname{Map}'(\mathbb{Z}_N, \mathfrak{g}) = \{f \in \operatorname{Map}(\mathbb{Z}_N, \mathfrak{g}) \mid \sum_{t \in \mathbb{Z}_N} f(t) \in \mathfrak{t}\}$ . Similar statements hold for  $\ker(\bar{L}^{(N)}(b))$

**Observation 3.7** We have<sup>24</sup>

$$\det(\hat{L}^{(N)}(b)) = \pm \det(1_{\mathfrak{k}} - \exp(\text{ad}(b))|_{\mathfrak{k}}) \cdot N^d, \quad (3.20)$$

$$\det(\check{L}^{(N)}(b)) = \pm \det(1_{\mathfrak{k}} - \exp(\text{ad}(b))|_{\mathfrak{k}}) \cdot N^d, \quad (3.21)$$

$$\det(\bar{L}^{(N)}(b)) = \det(1_{\mathfrak{k}} - \exp(\text{ad}(b))|_{\mathfrak{k}})^2 \left(\frac{N}{2}\right)^d 2^{2r} \quad \text{if } N \text{ is even} \quad (3.22)$$

where  $d := \dim(\text{Map}(\mathbb{Z}_N, \mathfrak{g})) = N \dim(\mathfrak{g})$ ,  $r := \dim(\mathfrak{t})$ .

*Proof.* Let us prove<sup>25</sup> Eq. (3.20). The proof of Eqs. (3.21) and (3.22) (and Eq. (3.24) below) is similar. First observe that

$$\det(\hat{L}^{(N)}(b)) = \det(N(\tau_1 e^{\text{ad}(b)/N} - 1)) = N^{d'} \det(\tau_1 - e^{-\text{ad}(b)/N}) \quad (3.23)$$

where  $d' := \dim(\text{Map}'(\mathbb{Z}_N, \mathfrak{g})) = d - \dim(\mathfrak{t})$  and where we have used that  $e^{\text{ad}(b)/N}$  is orthogonal. The complexified operator  $(\tau_1 - e^{-\text{ad}(b)/N}) \otimes \text{id}_{\mathbb{C}}$  is diagonalizable with eigenvalues

$$\begin{aligned} \lambda_{k,\alpha} &:= e^{\frac{2\pi i k}{N}} - e^{\frac{-\alpha(b)}{N}}, \quad \text{for each } k \in \mathbb{Z}_N, \alpha \in \mathcal{R}_{\mathbb{C}} \\ \mu_{k,a} &:= e^{\frac{2\pi i k}{N}} - 1 \quad \text{for each } k \in (\mathbb{Z}_N \setminus \{0\}), a \in \{1, 2, \dots, \dim(\mathfrak{t})\} \end{aligned}$$

where  $\mathcal{R}_{\mathbb{C}}$  denotes the set of complex roots of  $\mathfrak{g}$  w.r.t.  $\mathfrak{t}$  (cf. part A of the Appendix). Using the two polynomial equations  $x^N - 1 = \prod_{k=0}^{N-1} (x - e^{\frac{2\pi i k}{N}}) = (-1)^N \prod_{k=0}^{N-1} (e^{\frac{2\pi i k}{N}} - x)$  and  $x^{N-1} + x^{N-2} + \dots + 1 = \prod_{k=1}^{N-1} (x - e^{\frac{2\pi i k}{N}}) = (-1)^{N-1} \prod_{k=1}^{N-1} (e^{\frac{2\pi i k}{N}} - x)$  we therefore obtain

$$\begin{aligned} \det(\tau_1 - e^{-\text{ad}(b)/N}) &= \det_{\mathbb{C}}((\tau_1 - e^{-\text{ad}(b)/N}) \otimes \text{id}_{\mathbb{C}}) \\ &= \left( \prod_{\alpha \in \mathcal{R}_{\mathbb{C}}} \prod_k \{e^{\frac{2\pi i k}{N}} - e^{\frac{-\alpha(b)}{N}}\} \right) \left( \prod_{a \leq \dim(\mathfrak{t})} \{ \prod_{k \neq 0} (e^{\frac{2\pi i k}{N}} - 1) \} \right) \\ &= \left( \prod_{\alpha \in \mathcal{R}_{\mathbb{C}}} (-1)^N \{e^{-\alpha(b)} - 1\} \right) \left( \prod_{a \leq r} (-1)^{N-1} \{N\} \right) = (-1)^{r(N-1)N^r} \prod_{\alpha \in \mathcal{R}_{\mathbb{C}}} \{e^{-\alpha(b)} - 1\} \end{aligned}$$

The assertion now follows by combining the last equation with Eq. (3.23) above and by taking into account the relations  $d = d' + r$  and  $\prod_{\alpha \in \mathcal{R}_{\mathbb{C}}} (e^{-\alpha(b)} - 1) = \prod_{\alpha \in \mathcal{R}_{\mathbb{C}}} (1 - e^{\alpha(b)}) = \det_{\mathbb{C}}((1_{\mathfrak{k}} - \exp(\text{ad}(b))|_{\mathfrak{k}}) \otimes \text{id}_{\mathbb{C}}) = \det(1_{\mathfrak{k}} - \exp(\text{ad}(b))|_{\mathfrak{k}})$ .

□

**Remark 3.8** On the other hand<sup>26</sup>

$$\det(\hat{\partial}_t + \text{ad}(b)) = \det(\check{\partial}_t + \text{ad}(b)) = \pm N^d \det(1_{\mathfrak{k}} - ((1 + \frac{\text{ad}(b)}{N})^N)|_{\mathfrak{k}}) \quad (3.24)$$

Eq. (3.24) shows that, when working with the operators  $\hat{\partial}_t + \text{ad}(b)$  and  $\check{\partial}_t + \text{ad}(b)$  instead of  $\hat{L}^{(N)}(b)$  and  $\check{L}^{(N)}(b)$ , we have no chance of arriving at the expression  $\det(1_{\mathfrak{k}} - \exp(\text{ad}(b))|_{\mathfrak{k}})$  without letting  $N \rightarrow \infty$ .

### 3.3.4 The operator $L^{(N)}(B)$

For all  $B \in \mathcal{B}(b\mathcal{K})$  we will denote by  $L^{(N)}(B)$  the operator  $\mathcal{A}^{\perp}(K) \rightarrow \mathcal{A}^{\perp}(K)$  which, under the identification

$$\mathcal{A}^{\perp}(K) \cong \text{Map}(\mathbb{Z}_N, C^1(K_1, \mathfrak{g})) \oplus \text{Map}(\mathbb{Z}_N, C^1(K_2, \mathfrak{g}))$$

<sup>24</sup>the  $-$  sign in Eqs. (3.20) and (3.21) holds iff both  $r$  and  $N - 1$  are odd

<sup>25</sup>We prefer to prove Eq. (3.20) directly instead of exploiting Eq. (3.19) above

<sup>26</sup>the explicit expression of  $\det(\bar{\partial}_t + \text{ad}(b))$  is more complicated and will not be relevant for us; we omit the corresponding explicit formula

is given by (cf. Remark 3.10 below)

$$L^{(N)}(B) = \begin{pmatrix} \hat{L}^{(N)}(B) & 0 \\ 0 & \check{L}^{(N)}(B) \end{pmatrix} \quad (3.25)$$

Here  $\hat{L}^{(N)}(B) : \text{Map}(\mathbb{Z}_N, C^1(K_1, \mathfrak{g})) \rightarrow \text{Map}(\mathbb{Z}_N, C^1(K_1, \mathfrak{g}))$  and  $\check{L}^{(N)}(B) : \text{Map}(\mathbb{Z}_N, C^1(K_2, \mathfrak{g})) \rightarrow \text{Map}(\mathbb{Z}_N, C^1(K_2, \mathfrak{g}))$  are given by

$$\hat{L}^{(N)}(B) \cong \oplus_{\bar{e} \in \mathfrak{F}_0(K_1|K_2)} \hat{L}^{(N)}(B(\bar{e})) \quad (3.26a)$$

$$\check{L}^{(N)}(B) \cong \oplus_{\bar{e} \in \mathfrak{F}_0(K_1|K_2)} \check{L}^{(N)}(B(\bar{e})) \quad (3.26b)$$

where  $\mathfrak{F}_0(K_1|K_2)$  is as in Eq. (3.2) above. In Eqs. (3.26a) and (3.26b) we used the obvious identification

$$\text{Map}(\mathbb{Z}_N, C^1(K_j, \mathfrak{g})) \cong \oplus_{e \in \mathfrak{F}_0(K_j)} \text{Map}(\mathbb{Z}_N, \mathfrak{g}) \cong \oplus_{\bar{e} \in \mathfrak{F}_0(K_1|K_2)} \text{Map}(\mathbb{Z}_N, \mathfrak{g})$$

Observe that  $L^{(N)}(B)$  leaves the subspace  $\check{\mathcal{A}}^\perp(K)$  of  $\mathcal{A}^\perp(K)$  invariant. The restriction of  $L^{(N)}(B)$  onto  $\check{\mathcal{A}}^\perp(K)$  will also be denoted by  $L^{(N)}(B)$  in the following.

### Observation 3.9

- i) The operator  $S^{(N)}(B) := \star_K L^{(N)}(B)$  is symmetric w.r.t. the scalar product  $\ll \cdot, \cdot \gg_{\check{\mathcal{A}}^\perp(K)}$ .
- ii) From Eqs. (3.25), (3.26a), (3.26b), and Observation 3.7 above it follows that

$$\det(L^{(N)}(B)) = N^d \prod_{\bar{e} \in \mathfrak{F}_0(K_1|K_2)} \det(1_{\mathfrak{t}} - \text{ad}(B(\bar{e}))|_{\mathfrak{t}})^2 \quad (3.27)$$

where  $d := \dim(\mathcal{A}^\perp(K))$

**Remark 3.10** Recall that in the definition of the operator  $L^{(N)}(B)$  above the operators  $\hat{L}^{(N)}(b)$  and  $\check{L}^{(N)}(b)$  appeared on the RHS of Eqs. (3.26a) and (3.26b). One might wonder why we did not make use of the operator  $\bar{L}^{(N)}(b)$  instead. This question is analogous to the question asked in Remark 5.3 in [30] and has a similar answer:

If we identify the space  $\mathcal{A}^\perp(K)$  above with the space  $\mathcal{A}_{\text{altern}}^\perp(K)$  which was introduced in Appendix D in [30] then an alternative ansatz for the operator  $S^{(N)}(B) = \star_K L^{(N)}(B)$  is suggested, and this alternative ansatz indeed involves the operators  $\bar{L}^{(N)}(b)$ .

For the purpose of the present paper it is sufficient to work with the original definition of  $S^{(N)}(B)$ . On the other hand, if links with crossing points are studied, cf. Sec. 6 below, then it may well turn out that the alternative definition for  $S^{(N)}(B)$  is superior to the one used above.

### 3.4 Comparison of Approach I and Approach II

As we mentioned at the beginning of Sec. 3 Approach II has several advantages over Approach I. In particular, the very artificial condition (Mod3) of Approach I is also eliminated in Approach II. Moreover, (Mod2) of Approach I does not appear in Approach II. On the other hand, Approach II has the following two disadvantages compared to Approach I:

- (D1) The map  $\star_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{B}(b\mathcal{K})$  introduced in Sec. 3.2.3 is less natural than the map  $\star_K : C^2(K_1, \mathfrak{t}) \oplus C^2(K_2, \mathfrak{t}) \rightarrow C^0(K_2, \mathfrak{t}) \oplus C^0(K_1, \mathfrak{t})$  given by Eq. (2.18). In particular,  $\star_{b\mathcal{K}}$  is not bijective<sup>27</sup>.

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<sup>27</sup>and, in fact, cannot be expected to be so since in general  $\dim(\mathcal{E}(b\mathcal{K})) \neq \dim(\mathcal{B}(b\mathcal{K}))$

(D2) The discrete realizations  $\check{\mathcal{A}}^\perp(K)$  and  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  (cf. Eq. (2.13e) and Eq. (3.4)) of the spaces  $\check{\mathcal{A}}^\perp$  and  $\mathcal{A}_{\Sigma, \mathfrak{t}}$  of Sec. 2.1 do not “fit together” well in the sense that there is no natural discrete version of  $\mathcal{A}^\perp$  such that a discrete analogue of the decomposition (2.3) above holds.

(By contrast, in Approach I we use the discretization  $\mathcal{A}_{\Sigma, \mathfrak{t}}(K)$  instead of  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$ . Clearly,  $\mathcal{A}_{\Sigma, \mathfrak{t}}(K) \cong \mathcal{A}_c^\perp(K)$  and  $\check{\mathcal{A}}^\perp(K)$ , do “fit together”, cf. Eq. (2.14) above.)

**Remark 3.11** Observe that (Mod2) in Approach I has some similarities with the issue in (D1):

(Mod2) deals with an extension procedure, namely the extension of the map  $B_j : \mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2) \rightarrow \mathfrak{t}$  to a map  $\bar{B}_j : \mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_1|K_2) \cup \mathfrak{F}_0(K_2) \rightarrow \mathfrak{t}$ . On the other hand the definition of  $\star_{b\mathcal{K}}$  used in Approach II involves a similar extension procedure, cf. Step 2 in Sec. 3.2.3 above. Both of these extension procedures are not totally natural<sup>28</sup>.

We will see later that, once we pass to the  $BF$ -point of view, the similarities between Approach I and Approach II become even greater, cf. “Alternative 2” in part E of the Appendix below.

For the purposes of the present paper points (D1) and (D2) above do not represent a serious problem<sup>29</sup>. However, both points could have serious negative consequences for the simplicial program (cf. Appendix F and Remark F.1 below) so it would be desirable to find a way to eliminate them. In part D of the Appendix below we will sketch two methods which should allow point (D2) to be eliminated successfully<sup>30</sup>.

## 4 Approach II: the remaining details

### 4.1 Three conventions

**Convention 2** In the following  $\sim$  will denote equality up to a multiplicative constant. This “constant” may depend on  $G$ ,  $N$ ,  $\mathcal{K}$ , and  $k$  but it will never depend on the link  $L$ .

**Convention 3** A Euclidean space  $(V, \langle \cdot, \cdot \rangle_V)$  will often be denoted simply by  $V$  if no confusion about the scalar product  $\langle \cdot, \cdot \rangle_V$  can arise.

**Convention 4** We will simply write  $\star$  for each of the four maps

$$\star_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \text{Map}(\Sigma, \mathfrak{t}) \quad (4.1a)$$

$$\star_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{B}(b\mathcal{K}) \quad (4.1b)$$

$$\star_{b\mathcal{K}} : C^2(b\mathcal{K}, \mathbb{R}) \rightarrow \text{Map}(\Sigma, \mathbb{R}) \quad (4.1c)$$

$$\star_{b\mathcal{K}} : C^2(b\mathcal{K}, \mathbb{R}) \rightarrow C^0(b\mathcal{K}, \mathbb{R}) \quad (4.1d)$$

introduced in Sec. 3.2.3 above.

Recall that  $\star$  also denotes the maps  $\star : C_p(K_1) \rightarrow C_{2-p}(K_2)$  and  $\star : C_p(K_2) \rightarrow C_{2-p}(K_1)$ ,  $p \in \{0, 1, 2\}$ , introduced in Sec. 4.2 in [30]. These two maps (in the special case  $p = 1$ ) will also appear below.

<sup>28</sup>and it was therefore somewhat misleading to state above that in Approach II condition (Mod2) is eliminated; in fact, the extension “issue” in (Mod2) is simply replaced by another extension “issue”

<sup>29</sup>recall that we only consider the problem of discretizing the torus gauge-fixed CS path integral rather than the original (= non-gauge fixed) CS path integral

<sup>30</sup>“successfully” in the sense that a version of Theorem 5.3 below can be derived after performing the relevant modifications suggested in part D of the Appendix

## 4.2 Definition of $S_{CS}^{disc}(\check{A}^\perp, B)$ and $S_{CS}^{disc}(A_c, E)$

As the discrete analogues of the continuum expressions  $S_{CS}(\check{A}^\perp, B)$  and  $S_{CS}(A_c, E)$  in Eq. (2.10) and Eq. (3.9) above we will use the expressions<sup>31</sup>

$$S_{CS}^{disc}(\check{A}^\perp, B) := \pi k \ll \check{A}^\perp, \star_K L^{(N)}(B) \check{A}^\perp \gg_{\check{A}^\perp(K)} \quad (4.2a)$$

for  $B \in \mathcal{B}(b\mathcal{K})$ ,  $\check{A}^\perp \in \check{\mathcal{A}}^\perp(K)$  and

$$S_{CS}^{disc}(A_c, E) := -2\pi k \ll A_c, \partial_{b\mathcal{K}} E \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})} \quad (4.2b)$$

for  $A_c \in \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$ ,  $E \in \mathcal{E}(b\mathcal{K})$ .

## 4.3 Definition of $\text{Hol}_l^{disc}(\check{A}^\perp, A_c, B; \mathfrak{h})$

Let  $\check{A}^\perp \in \check{\mathcal{A}}^\perp(K)$ ,  $A_c \in \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$ ,  $B \in \mathcal{B}(b\mathcal{K})$ ,  $\mathfrak{h} \in [\Sigma, G/T]$  and let  $l = (l^{(k)})_{k \leq n}$  be a discrete loop in  $K_1 \times \mathbb{Z}_N$  and  $l' = (l'^{(k)})_{k \leq n'}$  a discrete loop in  $K_2 \times \mathbb{Z}_N$ , which is a “framing” of  $l$  in the sense of Sec. 4.4 in [30]. Without loss of generality<sup>32</sup> we can assume that  $n' = n$ .

By  $(l_\Sigma^{(k)})_{k \leq n}$ ,  $(l'_{\Sigma^1}^{(k)})_{k \leq n}$ ,  $(l_{\Sigma^1}^{(k)})_{k \leq n}$ , and  $(l'_\Sigma{}^{(k)})_{k \leq n}$  we denote the “projected” discrete loops in  $K_1$ ,  $K_2$ , and  $\mathbb{Z}_N$ , cf. Sec. 5.3 in [30]. In complete analogy to Eq. (5.17) in Sec. 5.3 in [30] we set<sup>33</sup>

$$\begin{aligned} \text{Hol}_l^{disc}(\check{A}^\perp, A_c, B; \mathfrak{h}) &:= \prod_{k=1}^n \exp \left( \frac{1}{2} (\check{A}^\perp(\bullet l_{\Sigma^1}^{(k)})) (l_\Sigma^{(k)}) + \frac{1}{2} (\check{A}^\perp(\bullet l'_{\Sigma^1}{}^{(k)})) (l'_\Sigma{}^{(k)}) + \frac{1}{2} A_c(l_\Sigma^{(k)}) + \frac{1}{2} A_c(l'_\Sigma{}^{(k)}) \right. \\ &\quad \left. + \frac{1}{2} \left( \int_{l_\Sigma^{(k)}} A_{\text{sg}}(\mathfrak{h}) + \int_{l'_{\Sigma^1}{}^{(k)}} A_{\text{sg}}(\mathfrak{h}) \right) + \frac{1}{2} B(\bullet l_\Sigma^{(k)}) \cdot \frac{1}{N} \text{sgn}(l_{\Sigma^1}^{(k)}) + \frac{1}{2} B(\bullet l'_{\Sigma^1}{}^{(k)}) \cdot \frac{1}{N} \text{sgn}(l'_{\Sigma^1}{}^{(k)}) \right) \end{aligned} \quad (4.3)$$

where  $\text{sgn}(e) \in \{-1, 0, 1\}$  is given by  $\text{sgn}(e) = \pm 1$  if  $e \in \pm \mathfrak{F}_1(\mathbb{Z}_N) \subset C_1(\mathbb{Z}_N)$  and  $\text{sgn}(e) = 0$  if  $e$  is the “empty” edge  $0 \in C_1(\mathbb{Z}_N)$ .

In the expression  $\int_{l_\Sigma^{(k)}} A_{\text{sg}}(\mathfrak{h})$  (resp.  $\int_{l'_{\Sigma^1}{}^{(k)}} A_{\text{sg}}(\mathfrak{h})$ ) we consider  $l_\Sigma^{(k)}$  (resp.  $l'_{\Sigma^1}{}^{(k)}$ ) as a smooth curve  $c : [0, 1] \rightarrow \Sigma$  in the obvious way.

## 4.4 Definition of $\text{Det}_{FP}^{disc}(B)$

Let us now introduce a discrete analogue  $\text{Det}_{FP}^{disc}(B)$  of the heuristic expression  $\text{Det}_{FP}(B) = \det(1_\mathfrak{t} - \exp(\text{ad}(B)))|_\mathfrak{t}$  given by Eq. (2.9) above. Motivated by Eq. (5.34) in [30] and Eq. (3.1) above we make the ansatz

$$\text{Det}_{FP}^{disc}(B) := \prod_{x \in \mathfrak{F}_0(b\mathcal{K})} \det(1_\mathfrak{t} - \exp(\text{ad}(B(x))))|_\mathfrak{t}^{1/2} \quad (4.4)$$

for every  $B \in \mathcal{B}(b\mathcal{K})$ .

**Remark 4.1** Eq. (4.4) is analogous but in some sense more natural than Eq. (5.34) in [30] since now all vertices  $x \in \mathfrak{F}_0(b\mathcal{K})$  are “on equal footing”.

## 4.5 Discrete version of $\int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(\mathfrak{h}), \star E \rangle$

As the discrete analogue of the expression  $\int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(\mathfrak{h}), \star E \rangle$  in Eq. (3.6) (where  $E \in \Omega^2(\Sigma, \mathfrak{t})$ ) we will take

$$\int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(\mathfrak{h}), \star E \rangle \quad (4.5)$$

(where  $E \in \mathcal{E}(b\mathcal{K})$  and where  $\star$  is as in Eq. (4.1a)).

<sup>31</sup>recall the definition of the operator  $L^{(N)}(B)$  in Sec. 3.3.4 above and recall also that according to Sec. 3.3.1 and Sec. 3.3.2  $L^{(N)}(B)$  is a natural discrete analogue of the operator  $\partial/\partial t + \text{ad}(B)$  appearing in Eq. (2.10)

<sup>32</sup>it is always possible to add “empty edges” to  $l$  or to  $l'$ , if necessary

<sup>33</sup>according to Convention 1 in Sec. 3.1 the expressions  $A_c(l_\Sigma^{(k)})$  and  $A_c(l'_{\Sigma^1}{}^{(k)})$  are indeed well-defined

#### 4.6 Discrete version of $1_{C^\infty(\Sigma, \mathfrak{t}_{reg})}(B)$

Recall that in Approach I we fixed a family  $(1_{\mathfrak{t}_{reg}}^{(s)})_{s>0}$ , of elements of  $C^\infty_{\mathbb{R}}(\mathfrak{t})$ , with the following properties:

- $\text{Image}(1_{\mathfrak{t}_{reg}}^{(s)}) \subset [0, 1]$  and  $\text{supp}(1_{\mathfrak{t}_{reg}}^{(s)}) \subset \mathfrak{t}_{reg}$
- $1_{\mathfrak{t}_{reg}}^{(s)} \rightarrow 1_{\mathfrak{t}_{reg}}$  pointwise as  $s \rightarrow 0$
- Each  $1_{\mathfrak{t}_{reg}}^{(s)}$  is invariant under the operation of the affine Weyl group  $\mathcal{W}_{\text{aff}}$  on  $\mathfrak{t}$

For fixed  $s > 0$  and  $B \in \mathcal{B}(b\mathcal{K})$  we will now take the expression

$$\prod_x 1_{\mathfrak{t}_{reg}}^{(s)}(B(x)) := \prod_{x \in \mathfrak{F}_0(b\mathcal{K})} 1_{\mathfrak{t}_{reg}}^{(s)}(B(x)) \quad (4.6)$$

as the discrete analogue of  $1_{C^\infty(\Sigma, \mathfrak{t}_{reg})}(B)$ . Later we will let  $s \rightarrow 0$ .

#### 4.7 Discrete versions of the two Gauss-type measures in Eq. (3.6)

i) Let  $D\check{A}^\perp$  denote the (normalized) Lebesgue measure on  $\check{A}^\perp(K)$  (cf. Convention 3). According to Eq. (4.2a) the complex measure

$$\exp(iS_{CS}^{disc}(\check{A}^\perp, B))D\check{A}^\perp \quad (4.7)$$

is a centered oscillatory Gauss type measure on  $\check{A}^\perp(K)$ , which – according to Eq. (3.27) – is non-degenerate if  $B \in \mathcal{B}_{reg}(b\mathcal{K})$  where

$$\mathcal{B}_{reg}(b\mathcal{K}) := \{B \in \mathcal{B}(b\mathcal{K}) \mid B(x) \in \mathfrak{t}_{reg} \text{ for all } x \in \mathfrak{F}_0(b\mathcal{K})\} \quad (4.8)$$

From Example 3.4 in [30] and Eq. (3.27) above we obtain

$$\begin{aligned} Z_B^{disc} &:= \int_{\sim} \exp(iS_{CS}^{disc}(\check{A}^\perp, B))D\check{A}^\perp \\ &\sim \det(L^{(N)}(B))^{-1/2} \sim \prod_{\bar{e} \in \mathfrak{F}_0(K_1|K_2)} \det(1_{\mathfrak{t}} - \text{ad}(B(\bar{e}))|_{\mathfrak{t}})^{-1} \end{aligned} \quad (4.9)$$

For later use let us set for every  $B \in \mathcal{B}_{reg}(b\mathcal{K})$

$$d\nu_B^{disc} := \frac{1}{Z_B^{disc}} \exp(iS_{CS}^{disc}(\check{A}^\perp, B))D\check{A}^\perp \quad (4.10)$$

Clearly,  $d\nu_B^{disc}$  is a normalized centered non-degenerate oscillatory Gauss type measure.

ii) Let  $DA_c$  denote the (normalized) Lebesgue measure on  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  and  $DE$  the (normalized) Lebesgue measure on  $\mathcal{E}(b\mathcal{K})$  (cf. again Convention 3). According to Eq. (4.2b) above, the complex measure

$$\exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE) \quad (4.11)$$

is a centered oscillatory Gauss type measure on  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) \oplus \mathcal{E}(b\mathcal{K})$ .

For later use (cf. Step 2 of Sec. 5 below) let us set

$$d\nu^{disc} := \frac{1}{Z^{disc}} \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE), \quad \text{with} \quad (4.12a)$$

$$Z^{disc} := \int_{\sim} \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE), \quad (4.12b)$$

Clearly,  $d\nu^{disc}$  is a normalized centered oscillatory Gauss type measure but it is not non-degenerate. Indeed, if  $S = S_{d\nu^{disc}}$  is the symmetric linear operator on  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) \oplus \mathcal{E}(b\mathcal{K})$  which



is associated to  $d\nu^{disc}$  (cf. Definition 3.1 in [30]) then we have  $\ker(S) = \mathcal{A}_{closed}(b\mathcal{K}) \oplus \mathcal{E}_c(b\mathcal{K})$  where we have set

$$\mathcal{E}_c(b\mathcal{K}) := \{E \in \mathcal{E}(b\mathcal{K}) \mid E \text{ is constant}\} \cong \mathfrak{t} \quad (4.13)$$

$$\mathcal{A}_{closed}(b\mathcal{K}) := \{A_c \in \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) \mid d_{b\mathcal{K}}A_c = 0\}, \quad (4.14)$$

Observe that since  $\partial_{b\mathcal{K}}$  is dual to  $d_{b\mathcal{K}}$  (w.r.t. the scalar products  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})}$  and  $\ll \cdot, \cdot \gg_{\mathcal{E}(b\mathcal{K})}$ ) the space

$$\mathcal{A}_{coex}(b\mathcal{K}) := \{\partial_{b\mathcal{K}}E \mid E \in \mathcal{E}(b\mathcal{K})\} \subset \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) \quad (4.15)$$

is the orthogonal complement of  $\mathcal{A}_{closed}(b\mathcal{K})$  in  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$ . In particular, we have

$$\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) = \mathcal{A}_{closed}(b\mathcal{K}) \oplus \mathcal{A}_{coex}(b\mathcal{K}) \quad (4.16)$$

#### 4.8 Definition of $\text{WLO}_{rig}^{disc}(L)$ and $\text{WLO}_{rig}(L)$

For the rest of this paper we will fix a discrete link  $L = (l_1, l_2, \dots, l_m)$  in  $K_1 \times \mathbb{Z}_N$  with “colors”  $(\rho_1, \rho_2, \dots, \rho_m)$ ,  $m \in \mathbb{N}$ . Moreover, for each  $i \leq m$  we fix a framing  $l'_i$  in the sense of Sec. 4.4 in [30] (in particular, each  $l'_i$  is a discrete loop in  $K_2 \times \mathbb{Z}_N$ ).

Using the definitions of the previous subsections we then arrive at the following rigorous simplicial analogue  $\text{WLO}_{rig}^{disc}(L)$  of the heuristic expression  $\text{WLO}(L)$  in Eq. (3.6):

$$\begin{aligned} \text{WLO}_{rig}^{disc}(L) &:= \lim_{s \rightarrow 0} \sum_{h \in [\Sigma, G/T]} \int_{\sim} \left\{ \left( \prod_x 1_{reg}^{(s)}(B(x)) \right) \text{Det}_{FP}^{disc}(B) \right. \\ &\quad \times \left[ \int_{\sim} \prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l'_i}^{disc}(\check{A}^\perp, A_c, B; h)) \exp(iS_{CS}^{disc}(\check{A}^\perp, B)) D\check{A}^\perp \right] \Big\}_{|B=\star E} \\ &\quad \times \exp(-2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{sg}(h), \star E \rangle) \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE) \end{aligned} \quad (4.17)$$

Finally, let us set

$$\text{WLO}_{rig}(L) := \frac{\text{WLO}_{rig}^{disc}(L)}{\text{WLO}_{rig}^{disc}(\emptyset)} \quad (4.18)$$

where  $\emptyset$  is the “empty” link<sup>34</sup>.

**Remark 4.2** i) We could equally state our main result below in terms of  $\text{WLO}_{rig}^{disc}(L)$  rather than  $\text{WLO}_{rig}(L)$ , part iii) of Remark 5.4 below. For stylistic reasons we prefer  $\text{WLO}_{rig}(L)$ .

ii) Recall that according to Remark 3.2 above it is probably possible to justify a simplified version of the heuristic equation Eq. (3.6). If this is the case then, clearly, we are lead to an analogous simplification of the definition of  $\text{WLO}_{rig}^{disc}(L)$  above.

## 5 The main result

### 5.1 The special class of links considered

From now on we will restrict ourselves to the special case where the discrete link  $L = (l_1, l_2, \dots, l_m)$  fixed in Sec. 4.8 above fulfills the following two conditions (as in [30] we consider each loop  $l_i$  has a piecewise smooth loop in  $\Sigma$ ; moreover we set  $l_\Sigma^i := (l_i)_\Sigma$  and  $l_{S^1}^i := (l_i)_{S^1}$ ):

<sup>34</sup>in other words:  $\text{WLO}_{rig}^{disc}(\emptyset)$  is given by the expression which we get from the RHS of Eq. (4.17) after omitting the product  $\prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l'_i}^{disc}(\check{A}^\perp, A_c, B; h))$

(NCP) The link  $L$  has no crossing points, i.e. the projected loops  $l_\Sigma^1, l_\Sigma^2, \dots, l_\Sigma^m$  are non-intersecting Jordan loops in  $\Sigma$ .

(NH) Each  $l_\Sigma^i$  is null-homologous.

Figures 1–3 below show examples for links fulfilling both (NCP) and (NH)

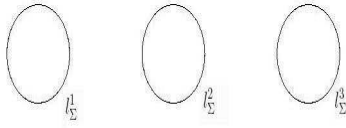


Figure 1:

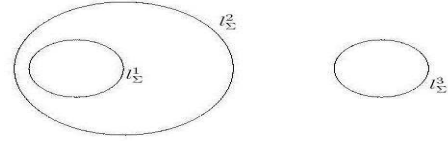


Figure 2:

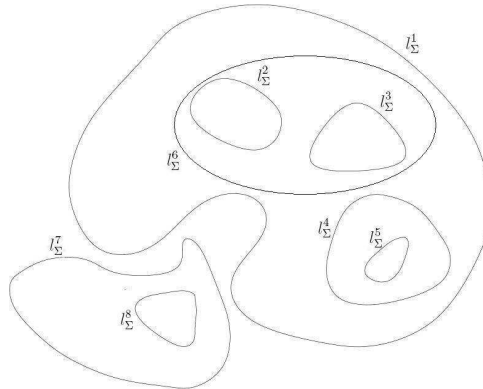


Figure 3:

**Remark 5.1** In spite of the simplicity of a link  $L$  fulfilling conditions (NCP) and (NH) the corresponding expression for the shadow invariant  $|L|$  is definitely non-trivial, cf. Eq. (B.9) in part B of the Appendix and Remark 5.4 below. Observe<sup>35</sup> in particular, that (NCP) (and (NH)) place no restrictions on the  $S^1$ -projections  $l_{S^1}^j$  of the loops  $l_j$ ,  $j \leq m$ , and so in general the link  $L$  will not be equivalent to a link with the property that there is a sequence  $(D_i)_{i \leq m}$  of pairwise disjoint disks  $D_i \subset \Sigma$  such that for each  $j$  the arc of  $l_\Sigma^j$  is contained in  $D_j$ . In particular, links fulfilling conditions (NCP) and (NH) will in general not be equivalent to a link consisting of only “vertical” loops, i.e. loops whose  $\Sigma$ -projections are “points”, cf. Fig. 4 for an example. (Vertical links are the only type of links appearing in [11, 12]).



Figure 4: A vertical link consisting of three loops

<sup>35</sup>for example this is the case for the links in Fig. 2 and Fig. 3 if, e.g.,  $l_{S^1}^j = i_{S^1}$  holds; by contrast the link in Fig. 1 will be equivalent to a vertical link if  $l_{S^1}^j = i_{S^1}$

## 5.2 The special class of framings considered

Recall that in Sec. 4.8 above we fixed a framing  $l'_i$  for each of the loops  $l_i$ ,  $i \leq m$ , appearing in the discrete link  $L = (l_1, l_2, \dots, l_m)$ .

From now on we will only consider the special case where the following conditions (FC1)-(FC4) are fulfilled<sup>36</sup> (here we use the short notation  $l_\Sigma^i := (l'_i)_\Sigma$ ,  $l_{S^1}^i := (l'_i)_{S^1}$ ):

- (FC1) For each  $i \leq m$  the loop  $l_\Sigma^i$  is null-homologous and intersects neither  $l_\Sigma^i$  nor itself, nor any of the loops  $l_\Sigma^j$ ,  $l_\Sigma^j$  with  $j \neq i$ .
- (FC2) For each  $i \leq m$  the loop  $l_\Sigma^i$  is “close” to  $l_\Sigma^i$  in the sense that the open region  $O_i \subset \Sigma$  “between”<sup>37</sup>  $\text{arc}(l_\Sigma^i)$  and  $\text{arc}(l_\Sigma^i)$  does not contain an element of  $\mathfrak{F}_0(b\mathcal{K})$ , cf. Example 5.2 below.
- (FC3) For each  $i \leq m$  the loop  $l_\Sigma^i$  has “the same orientation”<sup>38</sup> as  $l_\Sigma^i$ .
- (FC4) For simplicity we assume that  $l_{S^1}^i = l_{S^1}^i$  is fulfilled for each  $i \leq m$

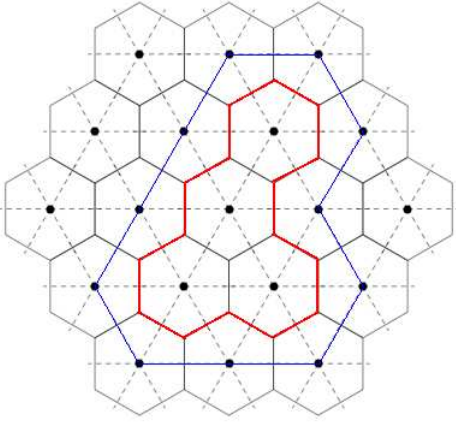


Figure 5:  $l_\Sigma^i$  is close to  $l_\Sigma^i$

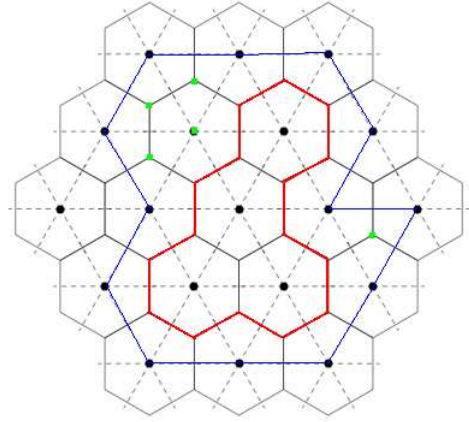


Figure 6:  $l_\Sigma^i$  is not close to  $l_\Sigma^i$

**Example 5.2** Suppose that  $K_1 = (\Sigma, \mathcal{C}_1)$  and  $K_2 = (\Sigma, \mathcal{C}_2)$  where  $\mathcal{C}_1$  is a “hexagonal” cell decomposition of  $\Sigma$  and  $\mathcal{C}_2$  is a “triangular” cell decomposition of  $\Sigma$  as in<sup>39</sup> Fig. 5 and Fig. 6. Assume that  $l_\Sigma^i$  is the red loop and  $l_\Sigma^i$  the blue loop. Accordingly, the set “ $O_i$ ” mentioned in (FC2) will be the open region between the blue and the red loop. Then  $l_\Sigma^i$  is “close” to  $l_\Sigma^i$  in Fig. 5 while in Fig. 6  $l_\Sigma^i$  is not “close” to  $l_\Sigma^i$  since  $O_i$  contains five elements of  $\mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2)$ , painted green<sup>40</sup>.

<sup>36</sup>from the assumptions (NCP) and (NH) on the link  $L$  it follows that one can usually find a tuple of framings fulfilling conditions (FC1)-(FC4). The only possible complication is that some of the loops  $l_i$ ,  $i \leq m$  come “very close” to each other. The aforementioned complication can always be resolved by replacing  $\mathcal{K}$  and  $\mathcal{K}'$  by “finer” polyhedral cell complexes and by considering the loops  $l_i$ ,  $l'_i$ ,  $i \leq m$  as discrete loops in the corresponding refined cell complexes

<sup>37</sup>more precisely, the unique connected component  $O$  of  $\Sigma \setminus (\text{arc}(l_\Sigma^i) \cup \text{arc}(l_\Sigma^i))$  such that  $\partial O = \text{arc}(l_\Sigma^i) \cup \text{arc}(l_\Sigma^i)$

<sup>38</sup>more precisely: the open region  $O_i$  lies either “to the right” of  $l_\Sigma^i$  and “to the left” of  $l_\Sigma^i$  or the other way around. Here “to the right/left” can, of course, be defined rigorously by using the orientation of  $\Sigma$  and the orientation of the loop  $l_\Sigma^i$  or  $l_\Sigma^i$ .

<sup>39</sup>as a side remark let us mention that in Fig. 5 and Fig. 6 we have omitted some of the edges of the corresponding barycentric subdivision  $b\mathcal{K}$ . In fact, the edges shown are exactly those of the coarser cell complex  $q\mathcal{K}$  mentioned in Remark 3.1 in Sec. 3.1 above

<sup>40</sup>in fact,  $O_i$  also contains five elements of  $\mathfrak{F}_0(K_1|K_2)$ , which are, however, not highlighted in Fig. 6

In the following we will take the set  $\mathcal{O}$  appearing in Sec. 3.2.3 to be

$$\mathcal{O} = \bigcup_i \mathcal{O}_i$$

Thus our definition of  $\star = \star_{b\mathcal{K}}$  will depend on the link  $L = (l_1, l_2, \dots, l_m)$  and the framings  $(l'_1, l'_2, \dots, l'_m)$ , cf. Remark 3.4 above.

**Assumption 1** *In the following we will assume for simplicity that the point  $\sigma_0 \in \Sigma$  fixed above (cf., e.g., Sec. 4.5) was chosen such that  $\sigma_0 \in \mathfrak{F}_0(b\mathcal{K})$  and  $\sigma_0 \notin \bigcup_i (\text{arc}(l_\Sigma^i) \cup \text{arc}(l_\Sigma'^i))$  hold.*

### 5.3 The main result

We are now ready to state our main result:

**Theorem 5.3** *Assume that the link  $L = (l_1, l_2, \dots, l_m)$  fixed above fulfills conditions (NCP) and (NH) above and that the framings  $(l'_1, l'_2, \dots, l'_m)$  fulfill conditions (FC1)-(FC4) above. Assume also that<sup>41</sup>  $k \geq c_{\mathfrak{g}}$  where  $c_{\mathfrak{g}}$  is the dual Coxeter number of  $\mathfrak{g}$ . Then  $\text{WLO}_{\text{rig}}(L)$  is well-defined and we have*

$$\text{WLO}_{\text{rig}}(L) = \frac{|L|}{|\emptyset|} \quad (5.1)$$

where  $\emptyset$  is the “empty link” and where  $|\cdot|$  is the shadow invariant associated to  $G$  and  $k$ , cf. part B of the Appendix.

**Remark 5.4** Theorem 5.3 can be considered<sup>42</sup> as a rigorous and more general version of the main result in [11]. Let us make the following remarks:

- i) In [11] only links composed of “vertical” loops  $l$ , cf. Remark 5.1 above, were considered (at a heuristic level). The class of links considered in Theorem 5.3 above is considerably larger<sup>43</sup>. On the other hand, links fulfilling conditions (NCP) and (NH) are still rather simple so one might wonder why we care about them. The reason why such links are still interesting is that the expression for the shadow invariant  $|L|$  for such links is quite complicated, cf. Eq. (B.9) in part B of the Appendix below. In particular, three of the four factors  $|L|_1^\varphi$ ,  $|L|_2^\varphi$ ,  $|L|_3^\varphi$ ,  $|L|_4^\varphi$  appearing in the formula for the shadow invariant of a general link  $L$  (cf. Eq. (B.4)) also appear in Eq. (B.9).

The fact that we have derived the RHS of Eq. (B.9) from the CS path integral is (hopefully) interesting by itself but, of course, we are mainly interested in the computation of  $\text{WLO}_{\text{rig}}(L)$  for general links, cf. Sec. 6 below for some comments in that direction.

- ii) Let us also emphasize that  $|L|$  here is really the shadow invariant associated to  $G$  and  $k$  and *not* to  $G$  and  $k + c_{\mathfrak{g}}$  where  $c_{\mathfrak{g}}$  is the dual Coxeter number of  $\mathfrak{g}$ . In other words: we do not have a “shift in  $k$ ” as predicted in much of the physicist literature, including [11, 12]. In Remark B.2 in part B of the Appendix below we will comment on this issue.

- iii) The explicit expression for  $\text{WLO}_{\text{rig}}^{\text{disc}}(L)$  is

$$\text{WLO}_{\text{rig}}^{\text{disc}}(L) = c_1 k^{c_2} \left( \prod_{\alpha \in \mathcal{R}_+} \sin\left(\frac{\langle \rho, \alpha \rangle}{k}\right) \right)^{\chi(\Sigma)} |L| \quad (5.2)$$

where  $\mathcal{R}_+$  and  $\rho$  are as in part A of the Appendix and where  $c_1, c_2 \in \mathbb{C}$  only depend on  $G, \mathcal{K}$ , and  $N$  but not on  $k$ . (We omit the precise formulas for  $c_1, c_2$ ).

<sup>41</sup>the situation  $0 < k < c_{\mathfrak{g}}$  is not interesting since in this case the set  $\Lambda_+^k$  appearing in Eq. (B.3) of part B of the Appendix below is empty, cf. Remark B.1 below. Accordingly,  $|L| = |\emptyset| = 0$ . It turns out that we then also have  $\text{WLO}_{\text{rig}}^{\text{disc}}(L) = \text{WLO}_{\text{rig}}^{\text{disc}}(\emptyset) = 0$

<sup>42</sup>if one ignores the issue of the “shift in  $k$ ”, cf. part ii) of the present Remark and Remark B.2 in part B of the Appendix

<sup>43</sup>also the corresponding set of equivalence classes is larger, cf. Remark 5.1 above

**Remark 5.5** One might wonder why we do not work with unframed loops, i.e. why we do not define  $\text{Hol}_l^{\text{disc}}(\check{A}^\perp, A_c, B; \mathbf{h})$  in analogy to Eq. (5.14) in [30] instead of Eq. (4.3) above. Also then we would obtain an analogue of Theorem 5.3 and the proof would be simpler than for the original Theorem 5.3.

The problem with unframed loops is that their crossing points will go “undetected”: there is no chance of obtaining a factor like  $|L|_4^\varphi$  in Eq. (B.4) below in the explicit formula for  $\text{WLO}_{\text{rig}}^{\text{disc}}(L)$ . So when working with unframed links we cannot hope to be able to find a generalization of Theorem 5.3 which includes the case of general links.

## 5.4 Proof of Theorem 5.3

Recall that Theorem 5.3 states that in the special situation described above  $\text{WLO}_{\text{rig}}(L)$  is well-defined and has the value  $|L|/|\emptyset|$ . In the following we will concentrate on the “computational half” of this statement. That  $\text{WLO}_{\text{rig}}(L)$  is well-defined in the first place will also become clear during the computations<sup>44</sup> even though we will rarely make explicit statements in this directions.

**Convention 5** *Without loss of generality<sup>45</sup> let us assume that the discrete loops  $l_i$ ,  $i \leq m$ , appearing above have the same length  $n \in \mathbb{N}$ .*

*Let us now introduce the following notation<sup>46</sup> for  $l = l_i = (l_i^{(k)})_{k \leq n}$ , and  $l' = l'_i = (l'_i{}^{(k)})_{k \leq n}$ ,  $i \in \{1, 2, \dots, m\}$  (we use Convention 1 above):*

$$l_\Sigma^{(k)} := \frac{1}{2} l_\Sigma^{(k)} + \frac{1}{2} l'_\Sigma{}^{(k)} \in C_1(K) \subset C_1(b\mathcal{K}), \quad k \leq n \quad (5.3)$$

$$l_\Sigma := \sum_{k=1}^n l_\Sigma^{(k)} \in C_1(K) \subset C_1(b\mathcal{K}) \quad (5.4)$$

$$A_{\text{sg}}(\mathbf{h})(l_\Sigma^{(k)}) := \frac{1}{2} \int_{l_\Sigma^{(k)}} A_{\text{sg}}(\mathbf{h}) + \frac{1}{2} \int_{l'_\Sigma{}^{(k)}} A_{\text{sg}}(\mathbf{h}), \quad k \leq n \quad (5.5)$$

$$A_{\text{sg}}(\mathbf{h})(l_\Sigma) := \frac{1}{2} \int_{l_\Sigma} A_{\text{sg}}(\mathbf{h}) + \frac{1}{2} \int_{l'_\Sigma} A_{\text{sg}}(\mathbf{h}) \quad (5.6)$$

$$A_c(l_\Sigma^{(k)}) := \frac{1}{2} A_c(l_\Sigma^{(k)}) + \frac{1}{2} A_c(l'_\Sigma{}^{(k)}), \quad k \leq n, \quad A_c \in \mathcal{A}_{\Sigma, \mathbf{t}}(b\mathcal{K}) \quad (5.7)$$

### 5.4.1 Step 1: Performing the $\int_\sim \cdots \exp(iS_{CS}^{\text{disc}}(\check{A}^\perp, B)) D\check{A}^\perp$ integration in Eq. (4.17)

**Lemma 1** *Under the assumptions on the link  $L = (l_1, \dots, l_m)$  and the framings  $(l'_1, \dots, l'_m)$  made above we have for every fixed  $A_c \in \mathcal{A}_{\Sigma, \mathbf{t}}(b\mathcal{K})$ ,  $B \in \mathcal{B}_{\text{reg}}(b\mathcal{K})$ , and  $\mathbf{h} \in [\Sigma, G/T]$*

$$\int_\sim \prod_i \text{Tr}_{\rho_i}(\text{Hol}_{l_i}^{\text{disc}}(\check{A}^\perp, A_c, B; \mathbf{h})) \exp(iS_{CS}^{\text{disc}}(\check{A}^\perp, B)) D\check{A}^\perp = Z_B^{\text{disc}} \prod_i \text{Tr}_{\rho_i}(\text{Hol}_{l_i}^{\text{disc}}(0, A_c, B; \mathbf{h})) \quad (5.8)$$

where  $Z_B^{\text{disc}} = \int_\sim \exp(iS_{CS}^{\text{disc}}(\check{A}^\perp, B)) D\check{A}^\perp$

*Proof.* Let  $A_c \in \mathcal{A}_{\Sigma, \mathbf{t}}(b\mathcal{K})$ ,  $B \in \mathcal{B}_{\text{reg}}(b\mathcal{K})$ , and  $\mathbf{h} \in [\Sigma, G/T]$  be as in the assertion of the lemma. In order to prove the lemma we will apply Proposition 3.8 (and Remark 3.9) in [30] to the special situation where

- $V = \check{\mathcal{A}}^\perp(K)$ ,
- $d\mu = d\nu_B^{\text{disc}}$  with  $d\nu_B^{\text{disc}} = \frac{1}{Z_B^{\text{disc}}} \exp(iS_{CS}^{\text{disc}}(\check{A}^\perp, B)) D\check{A}^\perp$  (cf. Sec. 4.7 above),

<sup>44</sup>in order to check well-definedness we should, of course, reverse the order of our considerations/computations: we first check that the expressions appearing in Step 6 are well-defined. Based on this we can verify that also the expressions in Step 5 must be well-defined and so on until we arrive at the expressions in Step 1

<sup>45</sup>observe that it is always possible to insert empty edges if necessary. Recall also that for each  $i \leq m$  the framed loop  $l'_i$  has the same length as  $l_i$  (cf. Sec. 4.3 above)

<sup>46</sup>the notation  $l_\Sigma$  introduced in (5.4) and (5.6) will be useful from Step 2 on, i.e. after the non-Abelian fields have been integrated out

- $(Y_k^{i,a})_{i \leq m, k \leq n, a \leq \dim(\mathfrak{g})}$  is the family of maps  $Y_k^{i,a} : \check{\mathcal{A}}^\perp(K) \rightarrow \mathbb{R}$  given by (cf. Convention 5 above)

$$Y_k^{i,a}(\check{A}^\perp) := \left\langle T_a, (\check{A}^\perp(\bullet l_{S^1}^{i(k)}))(\mathfrak{l}_\Sigma^{i(k)}) + A_c(\mathfrak{l}_\Sigma^{i(k)}) + A_{\text{sg}}(\mathfrak{h})(\mathfrak{l}_\Sigma^{i(k)}) \right. \\ \left. + \frac{1}{2}B(\bullet l_\Sigma^{i(k)}) \cdot \frac{1}{N} \text{sgn}(l_{S^1}^{i(k)}) + \frac{1}{2}B(\bullet l_\Sigma^{i(k)}) \cdot \frac{1}{N} \text{sgn}(l_{S^1}^{i(k)}) \right\rangle \quad (5.9)$$

where  $(T_a)_{a \leq \dim(\mathfrak{g})}$  is an arbitrary  $\langle \cdot, \cdot \rangle$ -ONB of  $\mathfrak{g}$  (which will be kept fixed in the following), and

- $\Phi : \mathbb{R}^{m \times n \times \dim(\mathfrak{g})} \rightarrow \mathbb{C}$  is given by

$$\Phi((x_k^{i,a})_{i,k,a}) = \prod_i \text{Tr}_{\rho_i} \left( \prod_k \exp \left( \sum_a T_a x_k^{i,a} \right) \right) \quad \text{for all } (x_k^{i,a})_{i,k,a} \in \mathbb{R}^{m \times n \times \dim(\mathfrak{g})} \quad (5.10)$$

Observe that

- i)  $d\nu_B^{disc}$  is a well-defined normalized non-degenerate centered oscillatory Gauss-type measure. This was observed already in Sec. 4.7 above.
- ii)  $\Phi \in \mathcal{P}_{exp}(\mathbb{R}^{m \times n \times \dim(\mathfrak{g})})$  since

$$\Phi((x_k^{i,a})_{i,k,a}) = \prod_i \text{Tr}_{\rho_i} \left( \prod_k \exp \left( \sum_a T_a x_k^{i,a} \right) \right) = \prod_i \text{Tr}_{\text{End}(V_i)} \left( \prod_k \rho_i \left( \exp \left( \sum_a T_a x_k^{i,a} \right) \right) \right) \\ = \prod_i \text{Tr}_{\text{End}(V_i)} \left( \prod_k \exp_{\text{End}(V_i)} \left( \sum_a ((\rho_i)_* T_a) x_k^{i,a} \right) \right) \quad (5.11)$$

where  $\exp_{\text{End}(V_i)}$  is exponential map of the associative algebra<sup>47</sup>  $\text{End}(V_i)$  and  $(\rho_i)_* : \mathfrak{g} \rightarrow \text{gl}(V_i)$ , for  $i \leq m$ , is the Lie algebra representation induced by  $\rho_i$ .

- iii) For all  $i \leq m, k \leq n, a \leq \dim(\mathfrak{g})$  we have

$$\int_{\sim} Y_k^{i,a} d\nu_B^{disc} = Y_k^{i,a}(0) \quad (5.12)$$

In order to see this let us introduce  $j : \mathbb{Z}_N \rightarrow \mathcal{A}_{\Sigma, \mathbb{R}}(K)$  by  $j(t) := \begin{cases} \mathfrak{l}_\Sigma^{i(k)} & \text{if } t = \bullet l_{S^1}^{i(k)} \\ 0 & \text{if } t \neq \bullet l_{S^1}^{i(k)} \end{cases}$  and

set  $\check{j}_a := p(T_a j)$  where  $p : \mathcal{A}^\perp(K) \rightarrow \check{\mathcal{A}}^\perp(K)$  is the  $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp(K)}$ -orthogonal projection. Then

$$Y_k^{i,a}(\check{A}^\perp) - Y_k^{i,a}(0) = \langle T_a, ((\check{A}^\perp)(\bullet l_{S^1}^{i(k)}))(\mathfrak{l}_\Sigma^{i(k)}) \rangle = \ll \check{A}^\perp, T_a j \gg_{\mathcal{A}^\perp(K)} = \ll \check{A}^\perp, \check{j}_a \gg_{\check{\mathcal{A}}^\perp(K)}$$

On the other hand since  $d\nu_B^{disc}$  is centered Example 3.4 in [30] implies that  $\int_{\sim} \ll \cdot, \check{j}_a \gg_{\check{\mathcal{A}}^\perp(K)} d\nu_B^{disc} = 0$ . Since  $d\nu_B^{disc}$  is also normalized we obtain Eq. (5.12).

- iv) For all  $i, i' \leq m, k, k' \leq n, a, a' \leq \dim(\mathfrak{g})$  we have

$$\int_{\sim} Y_k^{i,a} Y_{k'}^{i',a'} d\nu_B^{disc} = \int_{\sim} Y_k^{i,a} d\nu_B^{disc} \int_{\sim} Y_{k'}^{i',a'} d\nu_B^{disc} \quad (5.13)$$

This follows from Eq. (5.12) above and

$$\int_{\sim} (Y_k^{i,a} - Y_k^{i,a}(0))(Y_{k'}^{i',a'} - Y_{k'}^{i',a'}(0)) d\nu_B^{disc} = \int_{\sim} \ll \cdot, \check{j}_a \gg_{\check{\mathcal{A}}^\perp(K)} \ll \cdot, \check{j}_{a'} \gg_{\check{\mathcal{A}}^\perp(K)} d\nu_B^{disc} \\ \stackrel{(*)}{\sim} \ll \check{j}_a, (\star_K L^{(N)}(B))^{-1} \check{j}_{a'} \gg_{\check{\mathcal{A}}^\perp(K)} \stackrel{(**)}{=} 0 \quad (5.14)$$

<sup>47</sup> observe that the vector spaces underlying  $\text{End}(V_i)$  and  $\text{gl}(V_i)$  coincide

where  $\check{j}_a$  is as in point iii) above and where  $\check{j}'_{a'}$  is defined in a completely analogous way with  $i, k$ , and  $a$  replaced by  $i', k'$ , and  $a'$ . Here step (\*) follows from Example 3.4 in [30] and step (\*\*) follows since according to assumption (FC1) on the framings we have

$$\star l_{\Sigma}^{i_1(k_1)} \neq l_{\Sigma}^{i_2(k_2)} \quad (5.15)$$

for all  $i_1, i_2 \leq m$ , and  $k_1, k_2 \leq n$

Thus the assumptions of Proposition 3.8 in [30] are fulfilled and we obtain

$$\begin{aligned} & \frac{1}{Z_B^{disc}} \int_{\sim} \prod_i \text{Tr}_{\rho_i}(\text{Hol}_i^{disc}(\check{A}^{\perp}, A_c, B; \mathbf{h})) \exp(iS_{CS}^{disc}(\check{A}^{\perp}, B)) D\check{A}^{\perp} \\ &= \int_{\sim} \prod_i \text{Tr}_{\rho_i}(\prod_k \exp(\sum_a T_a Y_k^{i,a})) d\nu_B^{disc} = \int_{\sim} \Phi((Y_k^{i,a})_{i,k,a}) d\nu_B^{disc} \stackrel{(*)}{=} \Phi((\int_{\sim} Y_k^{i,a} d\nu_B^{disc})_{i,k,a}) \\ &= \Phi((Y_k^{i,a}(0))_{i,k,a}) = \prod_i \text{Tr}_{\rho_i}(\prod_k \exp(\sum_a T_a Y_k^{i,a}(0))) = \prod_i \text{Tr}_{\rho_i}(\text{Hol}_i^{disc}(0, A_c, B; \mathbf{h})) \quad (5.16) \end{aligned}$$

(Step (\*) follows from Proposition 3.8 and Remark 3.9 in [30]).

□

Using Lemma 1 and taking into account the implication

$$\prod_x 1_{\mathfrak{t}_{reg}}^{(s)}(B(x)) \neq 0 \quad \Rightarrow \quad B \in \mathcal{B}_{reg}(b\mathcal{K})$$

for all  $s > 0$  we now obtain from Eq. (4.17)

$$\begin{aligned} & \text{WLO}_{rig}^{disc}(L) \\ &= \lim_{s \rightarrow 0} \sum_{\mathbf{h} \in [\Sigma, G/T]} \int_{\sim} \left\{ \left( \prod_x 1_{\mathfrak{t}_{reg}}^{(s)}(B(x)) \right) \prod_i \text{Tr}_{\rho_i}(\text{Hol}_i^{disc}(0, A_c, B; \mathbf{h})) \text{Det}^{disc}(B) \right\}_{|B=\star E} \\ & \quad \times \exp(-2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(\mathbf{h}), \star E \rangle) \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE) \quad (5.17) \end{aligned}$$

where we have set

$$\text{Det}^{disc}(B) := \text{Det}_{FP}^{disc}(B) Z_B^{disc} \quad (5.18)$$

#### 5.4.2 Step 2: Performing the $\int_{\sim} \cdots \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE)$ -integration in (5.17)

With the help of Proposition 3.12 in [30] let us now evaluate the  $\int_{\sim} \cdots \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE)$ -integral appearing in Eq. (5.17). In order to do so we first rewrite the integrand in Eq. (5.17) in such a way that it assumes the form of the integrand on the LHS of the formula appearing in Proposition 3.12 in [30]. In order to achieve this we will now exploit the fact that all of the remaining fields  $A_c, E, B$  and  $A_{\text{sg}}(\mathbf{h})$  in Eq. (5.17) take values in the *Abelian* Lie algebra  $\mathfrak{t}$ . For fixed  $A_c, B$ , and  $\mathbf{h}$  we can therefore rewrite  $\text{Hol}_i^{disc}(0, A_c, B; \mathbf{h})$  as an exponential of a sum, namely as

$$\text{Hol}_i^{disc}(0, A_c, B; \mathbf{h}) = \exp(\Phi_i(B; \mathbf{h}) + \sum_k A_c(l_{\Sigma}^{i(k)})) \quad (5.19)$$

where we have set for each  $i \leq m$

$$\Phi_i(B; \mathbf{h}) := \sum_k A_{\text{sg}}(\mathbf{h})(l_{\Sigma}^{i(k)}) + \frac{1}{2} B(\bullet l_{\Sigma}^{i(k)}) \cdot \frac{1}{N} \text{sgn}(l_{S^1}^{i(k)}) + \frac{1}{2} B(\bullet l_{\Sigma}^{i(k)}) \cdot \frac{1}{N} \text{sgn}(l_{S^1}^{i(k)}), \quad (5.20)$$

cf. Eq. (4.3) and Convention 5 above.



Moreover, since  $\text{Hol}_i^{disc}(0, A_c, B; \mathbf{h}) \in T$  we can replace in Eq. (5.17) the characters  $\chi_i := \text{Tr}_{\rho_i}$ ,  $i \leq m$ , by their restrictions  $\chi_i|_T$ . But  $\chi_i|_T$  is just a linear combination of global weights, more precisely, for every  $b \in \mathfrak{t}$  we have

$$\text{Tr}_{\rho_i}(\exp(b)) = \chi_i|_T(\exp(b)) = \sum_{\alpha \in \Lambda} m_{\chi_i}(\alpha) e^{2\pi i \langle \alpha, b \rangle} \quad (5.21)$$

where  $m_{\chi_i}(\alpha)$  the multiplicity of  $\alpha \in \Lambda$  as a weight in  $\chi_i$  (here  $\Lambda \subset \mathfrak{t}^* \cong \mathfrak{t}$  denotes the lattice of the real weights associated to the pair  $(\mathfrak{g}, \mathfrak{t})$ , cf. part A of the Appendix below). Combining Eqs. (5.19) – (5.21) we obtain<sup>48</sup>

$$\begin{aligned} & \prod_i \text{Tr}_{\rho_i}(\text{Hol}_i^{disc}(0, A_c, B; \mathbf{h})) \\ &= \prod_i \left( \sum_{\alpha_i \in \Lambda} m_{\chi_i}(\alpha_i) \cdot \exp(2\pi i \langle \alpha_i, \Phi_i(B; \mathbf{h}) \rangle) \cdot \exp(2\pi i \sum_k \langle \alpha_i, A_c(\mathfrak{l}_{\Sigma}^{i(k)}) \rangle) \right) \\ &= \sum_{\alpha_1, \alpha_2, \dots, \alpha_m \in \Lambda} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \left( \prod_i \exp(2\pi i \langle \alpha_i, \Phi_i(B; \mathbf{h}) \rangle) \right) \exp(2\pi i \ll A_c, \sum_i \alpha_i \cdot \mathfrak{l}_{\Sigma}^i \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})}) \end{aligned} \quad (5.22)$$

where  $\mathfrak{l}_{\Sigma}^i$  is given according to Convention 5 above, i.e.

$$\mathfrak{l}_{\Sigma}^i = \sum_k \mathfrak{l}_{\Sigma}^{i(k)} \in C_1(b\mathcal{K}) \quad (5.23)$$

Let us now set for each  $\mathbf{h} \in [\Sigma, G/T]$ ,  $s > 0$ ,  $(\alpha_i)_i := (\alpha_1, \alpha_2, \dots, \alpha_m) \in \Lambda^m$ , and  $E \in \mathcal{E}(b\mathcal{K})$ :

$$\bar{F}_{(\alpha_i)_i, \mathbf{h}}^{(s)}(E) := F_{(\alpha_i)_i, \mathbf{h}}^{(s)}(\star E) \exp(-2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(\mathbf{h}), \star E \rangle) \quad (5.24)$$

where the first  $\star$  is the map (4.1b) and the second  $\star$  is the map (4.1a) and where for each  $B \in \mathcal{B}(b\mathcal{K})$  we have set

$$F_{(\alpha_i)_i, \mathbf{h}}^{(s)}(B) := \left( \prod_x 1_{\mathfrak{t}_{reg}}^{(s)}(B(x)) \right) \left( \prod_i \exp(2\pi i \langle \alpha_i, \Phi_i(B; \mathbf{h}) \rangle) \right) \text{Det}^{disc}(B) \quad (5.25)$$

Then, according to Eq. (5.22), we can rewrite Eq. (5.17) as

$$\begin{aligned} \text{WLO}_{rig}^{disc}(L) &= \lim_{s \rightarrow 0} \sum_{(\alpha_i)_i \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{\mathbf{h} \in [\Sigma, G/T]} \\ &\times \int_{\sim} \bar{F}_{(\alpha_i)_i, \mathbf{h}}^{(s)}(E) \exp(2\pi i \ll A_c, \sum_i \alpha_i \cdot \mathfrak{l}_{\Sigma}^i \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})}) \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE) \end{aligned} \quad (5.26)$$

(Observe that there are only finitely many  $(\alpha_i)_i \in \Lambda^m$  for which the product  $\prod_i m_{\chi_i}(\alpha_i)$  does not vanish. Accordingly, the summation  $\sum_{(\alpha_i)_i \in \Lambda^m} (\prod_i m_{\chi_i}(\alpha_i)) \cdots$  above is a finite and we can interchange it with  $\sum_{\mathbf{h} \in [\Sigma, G/T]}$ ).

Let us now fix for a while  $s > 0$ ,  $\mathbf{h} \in [\Sigma, G/T]$ , and  $(\alpha_i)_i \in \Lambda^m$  and evaluate the corresponding  $\int_{\sim} \cdots$ -integral in Eq. (5.26). In order to do so we will apply Proposition 3.12 of [30] to the special situation where

- $V := \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) \oplus \mathcal{E}(b\mathcal{K})$
- $V_0 := \mathcal{A}_{closed}(b\mathcal{K}) \oplus \mathcal{E}_c(b\mathcal{K})$ ,  $V_1 := \mathcal{E}'(b\mathcal{K})$ ,  $V_2 := \mathcal{A}_{coex}(b\mathcal{K})$ ,

where  $\mathcal{A}_{closed}(b\mathcal{K})$ ,  $\mathcal{A}_{coex}(b\mathcal{K})$ , and  $\mathcal{E}_c(b\mathcal{K})$  are as at the end of Sec. 4.7 and where  $\mathcal{E}'(b\mathcal{K})$  is the orthogonal complement of  $\mathcal{E}_c(b\mathcal{K})$  in  $\mathcal{E}(b\mathcal{K})$

- $d\mu := d\nu^{disc}$  with  $d\nu^{disc} = \frac{1}{Z^{disc}} \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE)$  (cf. Sec. 4.7 above), and

<sup>48</sup>here we are a bit sloppy and use the letter  $i$  both for the multiplication index and the imaginary unit

- $F := \bar{F}_{(\alpha_i)_{i,h}}^{(s)} \circ p$  where  $p : V_0 \oplus V_1 \rightarrow \mathcal{E}(b\mathcal{K})$  is the obvious projection.

The following remarks show that the assumptions of Proposition 3.12 in [30] are indeed fulfilled:

- $d\nu^{disc}$  is a normalized centered oscillatory Gauss type measure on  $\mathcal{A}_{\Sigma,t}(b\mathcal{K}) \oplus \mathcal{E}(b\mathcal{K})$  of the form  $d\nu^{disc}(A_c, E) = \frac{1}{Z^{disc}} \exp(i \ll A_c, (-2\pi k \partial_{b\mathcal{K}})E \gg_{\mathcal{A}_{\Sigma,t}(b\mathcal{K})})(DA_c \otimes DE)$
- The function  $F$  is bounded and uniformly continuous
- The image of the operator  $\partial_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{A}_{\Sigma,t}(b\mathcal{K})$  is  $\mathcal{A}_{coex}(b\mathcal{K})$  and its kernel is  $\mathcal{E}_c(b\mathcal{K})$ . Since  $V_1 = \mathcal{E}'(b\mathcal{K})$  is a linear complement of  $\mathcal{E}_c(b\mathcal{K})$ ,

$$M := -2\pi k(\partial_{b\mathcal{K}})|_{\mathcal{E}'(b\mathcal{K})}$$

is a well-defined linear isomorphism  $\mathcal{E}'(b\mathcal{K}) \rightarrow \mathcal{A}_{coex}(b\mathcal{K}) = V_2$ .

- According to Assumption (NH) on the link  $L$  and Assumption (FC1) on the framings there is for each  $\mathfrak{l}_\Sigma^i \in C_1(b\mathcal{K})$  a  $D_i \in C_2(b\mathcal{K})$  (unique up to an additive constant  $c \in \mathbb{R}$ ) such that

$$\mathfrak{l}_\Sigma^i = \partial_{b\mathcal{K}} D_i \quad (5.27)$$

holds. Thus<sup>49</sup>

$$v := 2\pi \sum_i \alpha_i \cdot \mathfrak{l}_\Sigma^i = \partial_{b\mathcal{K}}(2\pi \sum_i \alpha_i \cdot D_i)$$

is an element of  $V_2 = \mathcal{A}_{coex}(b\mathcal{K})$

Applying Proposition 3.12 in [30] to the present situation we therefore obtain

$$\begin{aligned} & \frac{1}{Z^{disc}} \int_{\sim} \bar{F}_{(\alpha_i)_{i,h}}^{(s)}(E) \exp(2\pi i \ll A_c, \sum_i \alpha_i \cdot \mathfrak{l}_\Sigma^i \gg_{\mathcal{A}_{\Sigma,t}(b\mathcal{K})}) \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE) \\ &= \int_{\sim} F(A_c, E) \exp(i \ll A_c, v \gg_{\mathcal{A}_{\Sigma,t}(b\mathcal{K})}) d\nu^{disc}(A_c, E) \\ & \stackrel{(+)}{\sim} \int_{V_0}^{\sim} F(x_0 - M^{-1}v) dx_0 \stackrel{(*)}{\sim} \int_{\mathfrak{t}}^{\sim} \bar{F}_{(\alpha_i)_{i,h}}^{(s)}(b - M^{-1}v) db \end{aligned} \quad (5.28)$$

where the two improper integrals  $\int^{\sim} \cdots dx_0$  and  $\int^{\sim} \cdots db$  are defined<sup>50</sup> according to Convention 2 in [30]. In Step (+) we have applied Proposition 3.12 of [30]. Step (\*) above follows because  $F(x_0 - M^{-1}v)$  depends only on the  $\mathcal{E}_c(b\mathcal{K})$ -component of  $x_0 \in V_0 = \mathcal{A}_{closed}(b\mathcal{K}) \oplus \mathcal{E}_c(b\mathcal{K}) \cong \mathcal{A}_{closed}(b\mathcal{K}) \oplus \mathfrak{t}$ .

If  $D_i \in C_2(b\mathcal{K})$  is given by Eq. (5.27) and the normalization condition

$$\sum_{C \in \mathfrak{F}_2(b\mathcal{K})} \langle D_i, C \rangle_{C_2(b\mathcal{K})} = 0 \quad (5.29)$$

then we have  $\sum_i \alpha_i \cdot D_i \in \mathcal{E}'(b\mathcal{K})$ . From the definition of  $M$  we therefore obtain

$$M^{-1}v = -\frac{1}{k} \sum_i \alpha_i \cdot D_i \quad (5.30)$$

Combining Eqs. (5.26), (5.28), (5.24), and (5.30) we obtain

$$\begin{aligned} \text{WLO}_{rig}^{disc}(L) & \sim \lim_{s \rightarrow 0} \sum_{(\alpha_i)_{i \in \Lambda^m}} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{h \in [\Sigma, G/T]} \int_{\mathfrak{t}}^{\sim} db F_{(\alpha_i)_{i,h}}^{(s)}(B) \Big|_{B=b+\frac{1}{k} \sum_i \alpha_i \star D_i} \\ & \times \exp\left(-2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{sg}(h), b + \frac{1}{k} \sum_i \alpha_i \star D_i \rangle \right) \end{aligned} \quad (5.31)$$

<sup>49</sup>here we make the identification  $C_2(b\mathcal{K}) = C_2(b\mathcal{K}, \mathbb{R}) \cong C^2(b\mathcal{K}, \mathbb{R})$

<sup>50</sup>that the last (and therefore also the first) of these two improper integrals is in fact well-defined follows from Remark 3.13 in [30] and a “periodicity argument”, which will be given in Step 4 below

where the first  $\star$  is the map (4.1d) and the second  $\star$  is the map (4.1c).

Apart from the remaining  $\int_t^\sim \cdots db$ -integration (which will be taken care of in Step 4 below) we have now completed the evaluation of the  $\int_\sim \cdots \exp(iS_{CS}^{disc}(A_c, E))(DA_c \otimes DE)$ -integral in Eq. (5.17).

**Remark 5.6** It is not difficult to see<sup>51</sup> that Eq. (5.31) also holds if we (re)define  $D_i$  using the following normalization condition (instead of the normalization condition (5.29) above):

$$(\star D_i)(\sigma_0) = 0 \quad (5.32)$$

where  $\star$  is now the map (4.1c).

Since condition (5.32) is technically more convenient than (5.29) we will use the latter normalization condition in the following, i.e. we assume that  $D_i$  is given by Eqs. (5.27) and (5.32).

### 5.4.3 Step 3: Some simplifications

Before we replace  $F_{(\alpha_i)_{i,h}}^{(s)}(B)$  in Eq. (5.31) above by the RHS of Eq. (5.25) above let us first make some preparations. We begin with the following observation:

**Observation 5.7** *Let  $B \in \mathcal{B}(b\mathcal{K})$  be of the form*

$$B = b + \frac{1}{k} \sum_{i=1}^m \alpha_i \cdot \star D_i \quad (5.33)$$

*with  $b \in \mathfrak{t}$ ,  $\alpha_i \in \Lambda$  and  $D_i$  given by Eqs. (5.27) and (5.32) above. Then the map  $\mathfrak{F}_0(b\mathcal{K}) \ni \sigma \mapsto B(\sigma) \in \mathfrak{t}$  is constant on  $\text{arc}(l_\Sigma^j) \cap \mathfrak{F}_0(b\mathcal{K})$  and on  $\text{arc}(l_\Sigma^j) \cap \mathfrak{F}_0(b\mathcal{K})$ .*

**Remark 5.8** In fact, we have the following more general statement which will be also useful in Step 5 below: Let  $E := b + \frac{1}{k} \sum_{i=1}^m \alpha_i \cdot D_i \in \mathcal{E}(b\mathcal{K})$  and let  $\bar{B} := \overline{\star E} : \Sigma \rightarrow \mathfrak{t}$  where  $\overline{\star E}$  is given as in Eq. (3.14) above. Then  $\bar{B}$  is constant on each of the sets  $\text{arc}(l_\Sigma^j)$  and  $\text{arc}(l_\Sigma^j)$ ,  $j \leq m$ , and also on each of the connected components of  $\Sigma \setminus (\bigcup_j (\text{arc}(l_\Sigma^j) \cup \text{arc}(l_\Sigma^j)))$ .

Instead of giving a formal proof for the last claim it is probably more instructive to clarify things with the help of an example. For simplicity let us concentrate on the special case  $m = 1$  where  $E = b + \frac{1}{k} \alpha_1 D_1$ . In this case  $\bar{B}$  looks as in Fig. 7 below<sup>52</sup>, where the color of a point  $x$  indicates the value of  $\bar{B}(x)$ .

In Fig. 7 we assume that  $K_1$  is the hexagonal cell decomposition and  $K_2$  the triangular cell decomposition so the red loop in Fig. 7 is the loop  $l_\Sigma^1$  and the blue loop is the loop  $l_\Sigma^1$ .

In order to verify that  $\bar{B}$  really is given as in Fig. 7 observe that  $\mathcal{O} = O_1$  is simply the white region in Fig. 7. So according to Eq. (3.14) above we have  $\bar{B}(x) = \text{mean}_{\{F \in \mathfrak{F}_2(b\mathcal{K}) | x \in \overline{F}\}} E(F)$  if  $x \in \Sigma$  lies in the white area and  $\bar{B}(x) = \text{mean}_{\{F \in \mathfrak{F}_2(b\mathcal{K}) | x \in \overline{F}, F \text{ "is not white"}\}} E(F)$  otherwise. In both cases the arithmetic mean will actually be trivial (cf. Remark 3.4 above).

In order to make things even more concrete let us assume that  $\sigma_0$  happens to lie in the blue area. Then the color blue will correspond to the value  $b$ , and – depending on the orientations of the two loops (which are “coupled” to each other according to Condition (FC3) above) – either the color white indicates the value  $b + \frac{1}{2k} \alpha_1$  and the color red indicates the value  $b + \frac{1}{k} \alpha_1$  or (if the two loops have the inverse orientation) the color white indicates the value  $b - \frac{1}{2k} \alpha_1$  and the color red indicates the value  $b - \frac{1}{k} \alpha_1$ .

<sup>51</sup>this follows from Eq. (3.27) in Remark 3.13 in [30] and the periodicity properties of the integrand in  $\int_t^\sim \cdots db$  (for fixed  $h$  and  $\alpha_1, \dots, \alpha_m$ ), cf. Step 4 below

<sup>52</sup>for simplicity we have omitted some of the edges of  $\mathfrak{F}_1(b\mathcal{K})$ ; only those edges are shown which are contained in an edge in  $\mathfrak{F}_1(K_1)$  or  $\mathfrak{F}_2(K_1)$ ; similarly, each of the “faces” in Fig. 7 is actually a non-trivial union of 2-faces of  $b\mathcal{K}$

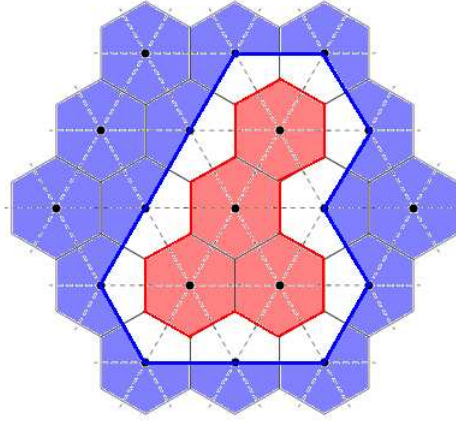


Figure 7:

Let us set

$$\sigma_i := \bullet l_{\Sigma}^{i(1)} \in \mathfrak{F}_0(K_1), \quad \sigma'_i := \bullet l'_{\Sigma}{}^{i(1)} \in \mathfrak{F}_0(K_2) \quad (5.34)$$

According to Observation 5.7 we have

$$B(\sigma_i) = B(\bullet l_{\Sigma}^{i(k)}) \quad \forall k \leq n \quad (5.35a)$$

$$B(\sigma'_i) = B(\bullet l'_{\Sigma}{}^{i(k)}) \quad \forall k \leq n \quad (5.35b)$$

for every  $B$  of the form in Eq. (5.33). Next observe that

$$\text{wind}(l_{S^1}^i) = \sum_k \frac{1}{N} \text{sgn}(l_{S^1}^{i(k)}), \quad \text{wind}(l'_{S^1}{}^i) = \sum_k \frac{1}{N} \text{sgn}(l'_{S^1}{}^{i(k)}) \quad (5.36)$$

where  $\text{wind}(l_{S^1}^i)$  is the winding number of  $l_{S^1}^i$  and  $\text{wind}(l'_{S^1}{}^i)$  is the winding number of  $l'_{S^1}{}^i$ . According to assumption (FC4) above in Sec. 5.2 we have

$$\epsilon_i := \text{wind}(l_{S^1}^i) = \text{wind}(l'_{S^1}{}^i) \quad (5.37)$$

Combining Eq. (5.31), Eq. (5.25), Eq. (5.20) with Eqs. (5.35a) – (5.37) we then obtain

$$\begin{aligned} \text{WLO}_{rig}^{disc}(L) &\sim \lim_{s \rightarrow 0} \sum_{(\alpha_i)_i \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{h \in [\Sigma, G/T]} \left( \prod_{j=1}^m \exp(2\pi i T_L^j(\alpha_i, h)) \right) \\ &\quad \times \int_t^\sim db \left( \exp(-2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{sg}(h), b \rangle) \right. \\ &\quad \left. [(\prod_x 1_{t_{reg}}^{(s)}(B(x))) (\prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle)) \text{Det}^{disc}(B)] \right]_{|B=b+\frac{1}{k} \sum_i \alpha_i \star D_i} \end{aligned} \quad (5.38)$$

where we have set for every  $j \leq m$ ,  $\alpha \in \Lambda$ , and  $h \in [\Sigma, G/T]$

$$T_L^j(\alpha, h) := \langle \alpha, A_{sg}(h)(l_{\Sigma}^j) \rangle - \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{sg}(h), \alpha \star D_j \rangle \quad (5.39)$$

where  $A_{sg}(h)(l_{\Sigma}^j) = \sum_k A_{sg}(h)(l_{\Sigma}^{j(k)})$ , cf. Convention 5 above.

**Lemma 2**  $T_L^i(\alpha, h) = 0$  for all  $i \leq m$ ,  $\alpha \in \Lambda$ , and  $h \in [\Sigma, G/T]$ .

*Proof.* Let  $l_\Sigma \in \{l_\Sigma^i, l_\Sigma'^i\}$  and let  $R$  be the (unique) connected component of  $\Sigma \setminus \text{arc}(l_\Sigma)$  with the property that  $l_\Sigma$  runs around  $R$  in the “positive” direction. From Stokes’ Theorem we have<sup>53</sup>

$$A_{\text{sg}}(\mathbf{h})(l_\Sigma) := \sum_k A_{\text{sg}}(\mathbf{h})(l_\Sigma^{(k)}) = \int_{l_\Sigma} A_{\text{sg}}(\mathbf{h}) \stackrel{(*)}{=} \int_{\Sigma \setminus \{\sigma_0\}} dA_{\text{sg}}(\mathbf{h}) \cdot (1_R - 1_R(\sigma_0)) \quad (5.40)$$

Let us identify  $l_\Sigma$  with an element of  $C_1(K_j) \subset C_1(b\mathcal{K})$  for  $j = 1$  resp.  $j = 2$  in the obvious way and let  $D \in C_2(b\mathcal{K})$  be given by  $\partial_{b\mathcal{K}} D = l_\Sigma$  with  $(\star D)(\sigma_0) = 0$ . Then we have

$$\star D = 1_R - 1_R(\sigma_0) \quad (5.41)$$

on  $\Sigma^{(2)}$  (whose complement in  $\Sigma$  is negligible in the integration appearing in Eq. (5.39) above). Applying Eqs. (5.40) and (5.41) both for  $l_\Sigma = l_\Sigma^i$  and  $l_\Sigma = l_\Sigma'^i$  and taking into account the definition of  $D_i$  (cf. Eq. (5.27) and Eq. (5.32) above) and Convention 5 at the beginning of Sec. 5.4 we now obtain indeed  $\langle \alpha, A_{\text{sg}}(\mathbf{h})(l_\Sigma^i) \rangle = \int_{\Sigma \setminus \{\sigma_0\}} \langle dA_{\text{sg}}(\mathbf{h}), \alpha \star D_i \rangle$ .  $\square$

Setting

$$F_{(\alpha_i)_i}^{(s)}(b) := \left[ \left( \prod_x 1_{\mathfrak{t}_{\text{reg}}}^{(s)}(B(x)) \right) \times \left( \prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma_j') \rangle) \text{Det}^{\text{disc}}(B) \right) \right]_{|B=b+\frac{1}{k} \sum_i \alpha_i \star D_i} \quad (5.42)$$

and taking into account Lemma 2 above and Remark 2.5 in [30], according to which  $n : [\Sigma, G/T] \rightarrow I = \ker(\exp|_{\mathfrak{t}}) \cong \mathbb{Z}^{\dim(\mathfrak{t})}$  given by

$$n(\mathbf{h}) = \int_{\Sigma \setminus \{\sigma_0\}} dA_{\text{sg}}(\mathbf{h}) \quad \forall \mathbf{h} \in [\Sigma, G/T] \quad (5.43)$$

is a well-defined bijection, we can rewrite Eq. (5.38) as

$$\text{WLO}_{\text{rig}}^{\text{disc}}(L) \sim \lim_{s \rightarrow 0} \sum_{(\alpha_i)_i \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{x \in I} \int^\sim db \, e^{-2\pi i k \langle x, b \rangle} F_{(\alpha_i)_i}^{(s)}(b) \quad (5.44)$$

#### 5.4.4 Step 4: Performing the remaining limit procedures $\int^\sim \cdots db$ , $\sum_{x \in I}$ , and $s \rightarrow 0$ in Eq. (5.44)

i) Let us first rewrite the  $\int^\sim \cdots db$  integral. The crucial observation is that for fixed  $x \in I$ ,  $s > 0$ , and  $(\alpha_i)_i \in \Lambda^m$  the function  $\mathfrak{t} \ni b \mapsto e^{-2\pi i k \langle x, b \rangle} F_{(\alpha_i)_i}^{(s)}(b) \in \mathbb{C}$  is invariant under all translations of the form  $b \mapsto b + y$  where  $y \in I = \ker(\exp|_{\mathfrak{t}}) \cong \mathbb{Z}^{\dim(\mathfrak{t})}$ . Indeed, for all  $b \in \mathfrak{t}$  and  $y \in I$  we have

$$1_{\mathfrak{t}_{\text{reg}}}^{(s)}(b + y) = 1_{\mathfrak{t}_{\text{reg}}}^{(s)}(b) \quad (5.45a)$$

$$e^{2\pi i \epsilon \langle \alpha, b+y \rangle} = e^{2\pi i \epsilon \langle \alpha, b \rangle} \quad \text{for all } \alpha \in \Lambda, \epsilon \in \mathbb{Z} \quad (5.45b)$$

$$\det(1_{\mathfrak{t}} - \exp(\text{ad}(b + y))|_{\mathfrak{t}}) = \det(1_{\mathfrak{t}} - \exp(\text{ad}(b))|_{\mathfrak{t}}) \quad (5.45c)$$

$$e^{-2\pi i k \langle x, b+y \rangle} = e^{-2\pi i k \langle x, b \rangle} \quad \text{for all } x \in I \quad (5.45d)$$

The first of these four equations follows from the assumptions in Sec. 4.6. The second equation follows because the assumption that  $G$  is simply-connected implies that

$$I = \Gamma \quad (5.46)$$

<sup>53</sup>Step  $(*)$  is evident for  $\sigma_0 \notin R$ . In the situation  $\sigma_0 \in R$  we apply Stokes theorem to the (unique) connected component  $R'$  in  $\Sigma \setminus \text{arc}(l_\Sigma)$  with the property that  $l_\Sigma$  runs around  $R'$  in the “negative” direction and use the fact that then  $-1_{R'}(\sigma) = 1_R(\sigma) - 1_R(\sigma_0)$  unless  $\sigma \in \text{arc}(l_\Sigma)$ .

where  $\Gamma \subset \mathfrak{t}$  is the lattice generated by the real coroots and, by definition,  $\Lambda$  is the lattice dual to  $\Gamma$ . The third equation follows from the second equation and the relations  $\mathcal{R} \subset \Lambda$  and<sup>54</sup>

$$\det(1_{\mathfrak{t}} - \exp(\text{ad}(b)))_{|\mathfrak{t}} = \prod_{\alpha \in \mathcal{R}} (1 - e^{2\pi i \langle \alpha, b \rangle})$$

where  $\mathcal{R}$  is the set of real roots of  $(\mathfrak{g}, \mathfrak{t})$ . Finally, in order to see that the fourth equation holds, observe that because of (5.46) it is enough to show that

$$\langle \check{\alpha}, \check{\beta} \rangle \in \mathbb{Z} \quad \text{for all coroots } \check{\alpha}, \check{\beta} \quad (5.47)$$

According to the general theory of semi-simple Lie algebras the entries of the so-called ‘‘Cartan matrix’’ are integers, i.e.  $2 \frac{\langle \check{\alpha}, \check{\beta} \rangle}{\langle \check{\alpha}, \check{\alpha} \rangle} \in \mathbb{Z}$ . Moreover, there are at most two different (co)roots lengths and the quotient between the square lengths of the long and short coroots is either 1, 2, or 3. Since the normalization of  $\langle \cdot, \cdot \rangle$  was chosen such that  $\langle \check{\alpha}, \check{\alpha} \rangle = 2$  holds if  $\check{\alpha}$  is a short coroot we therefore have  $\langle \check{\alpha}, \check{\alpha} \rangle / 2 \in \{1, 2, 3\}$  and (5.47) follows.

From Eqs. (5.42) and (5.45) (cf. also Eqs. (5.18), (4.4), and (4.9)) we conclude that  $\mathfrak{t} \ni b \mapsto e^{-2\pi i k \langle x, b \rangle} F_{(\alpha_i)_i}^{(s)}(b) \in \mathbb{C}$  is indeed  $I$ -periodic and we can therefore apply Eq. (3.27) in Remark 3.13 in [30] and obtain

$$\int^{\sim} db e^{-2\pi i k \langle x, b \rangle} F_{(\alpha_i)_i}^{(s)}(b) \sim \int_Q db e^{-2\pi i k \langle x, b \rangle} F_{(\alpha_i)_i}^{(s)}(b) \quad (5.48)$$

where on the RHS  $\int_Q \cdots db$  is now an ordinary integral and where we have set

$$Q := \left\{ \sum_i \lambda_i e_i \mid \lambda_i \in (0, 1) \text{ for all } i \leq m \right\} \subset \mathfrak{t}, \quad (5.49)$$

Here  $(e_i)_{i \leq m}$  is an (arbitrary) fixed basis of  $I$ .

According to Eq. (5.48) we can now rewrite Eq. (5.44) as

$$\text{WLO}_{rig}^{disc}(L) \sim \lim_{s \rightarrow 0} \sum_{(\alpha_i)_i \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{x \in I} \int_Q db e^{-2\pi i k \langle x, b \rangle} F_{(\alpha_i)_i}^{(s)}(b) \quad (5.50)$$

ii) We can now perform the infinite sum  $\sum_x$  and the  $\int \cdots db$ -integral in Eq. (5.50):

First recall that, due to Eq. (5.46) above and the definition of  $\Lambda$ ,  $\Lambda$  is dual to  $I$ . According to the (rigorous) Poisson summation formula for distributions we therefore have

$$\sum_{x \in I} e^{-2\pi i k \langle x, b \rangle} = \sum_{x \in \frac{1}{k}\Lambda} \delta_x(b) \quad (5.51)$$

where  $\delta_x$  is the delta distribution in  $x$ . Let us now apply Eq. (5.51) to the RHS of Eq. (5.50) above. In order to see that this is possible note first that not only  $F_{(\alpha_i)_i}^{(s)}$  is smooth but also the product  $1_Q F_{(\alpha_i)_i}^{(s)}$  because  $\partial Q \subset \mathfrak{t} \setminus \mathfrak{t}_{reg}$  and because  $F_{(\alpha_i)_i}^{(s)}$  vanishes on an open neighborhood of the set  $\mathfrak{t} \setminus \mathfrak{t}_{reg}$  (cf. the condition  $\text{supp}(1_{\mathfrak{t}_{reg}}^{(s)}) \subset \mathfrak{t}_{reg}$  in Sec. 4.6 and take into account that by Assumption 1 we have  $\sigma_0 \in \mathfrak{F}_0(bK)$  so, according to the definition of  $F_{(\alpha_i)_i}^{(s)}$  and Eq. (5.32), there is a factor  $1_{\mathfrak{t}_{reg}}^{(s)}(b)$  appearing in  $F_{(\alpha_i)_i}^{(s)}(b)$ ). Moreover, since  $Q$  is bounded  $1_Q F_{(\alpha_i)_i}^{(s)}$  has compact support. Thus we can indeed apply Eq. (5.51) to the RHS of Eq. (5.50) above and we then obtain

$$\text{WLO}_{rig}^{disc}(L) \sim \lim_{s \rightarrow 0} \sum_{(\alpha_i)_i \in \Lambda^m} \left( \prod_i m_{\chi_i}(\alpha_i) \right) \sum_{b \in \frac{1}{k}\Lambda} 1_Q(b) F_{(\alpha_i)_i}^{(s)}(b) \quad (5.52)$$

<sup>54</sup>observe that in contrast to the proof of Observation 3.7 in Sec. 3.3.3 above we now work with the set  $\mathcal{R}$  of real roots instead of the set of complex roots  $\mathcal{R}_{\mathbb{C}} = \{2\pi i \alpha \mid \alpha \in \mathcal{R}\}$

iii) Finally, let us also perform the  $s \rightarrow 0$  limit. Taking into account that  $1_{\text{treg}}^{(s)} \rightarrow 1_{\text{treg}}$  pointwise we obtain from Eq. (5.52) and Eq. (5.42) after the change of variable  $b \rightarrow kb =: \alpha_0$

$$\begin{aligned} \text{WLO}_{\text{rig}}^{\text{disc}}(L) &\sim \sum_{\alpha_0, \alpha_1, \dots, \alpha_m \in \Lambda} 1_{kQ}(\alpha_0) \left( \prod_{i=1}^m m_{\chi_i}(\alpha_i) \right) \\ &\times \left[ \left( \prod_x 1_{\text{treg}}(B(x)) \right) \prod_{j=1}^m \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) \text{Det}^{\text{disc}}(B) \right]_{|B=\frac{1}{k}(\alpha_0 + \sum_i \alpha_i \cdot \star D_i)} \end{aligned} \quad (5.53)$$

#### 5.4.5 Step 5: Rewriting $\text{Det}^{\text{disc}}(B)$ in Eq. (5.53)

In Steps 1–4 we have reduced the original “path integral” expression for  $\text{WLO}_{\text{rig}}(L)$  to a “finite” expression, i.e. an expression which does not involve any limit processes. Let us now have a closer look at  $\text{Det}^{\text{disc}}(B)$ , cf. Eq. (5.18) in Step 1 above. Before we do so we need some preparations.

Recall that  $O_j \subset \Sigma$ , for  $j \leq m$ , denotes the open “region between”  $\text{arc}(l_\Sigma^j)$  and  $\text{arc}(l_\Sigma'^j)$ , cf. Example 5.2 in Sec. 5.2 above.

From assumptions (NCP) and (NH) above it now follows that the set of connected components of  $\Sigma \setminus (\bigcup_j \text{arc}(l_\Sigma^j))$  has exactly  $m+1$  elements, which we will denote by  $Y_0, Y_1, \dots, Y_m$  in the following. From assumptions (FC1) and (FC2) it follows that also the set of connected components of  $\Sigma \setminus \bigcup_j \overline{O_j} = \Sigma \setminus \bigcup_j (O_j \cup \text{arc}(l_\Sigma^j) \cup \text{arc}(l_\Sigma'^j))$  has exactly  $m+1$  elements, which we will denote by  $Z_0, Z_1, \dots, Z_m$ . In the following we assume without loss of generality that the numeration of the  $Z_i$ ,  $i \leq m$ , was chosen such that

$$Z_i \subset Y_i \quad \forall i \in \{0, 1, \dots, m\}$$

holds. For later use we remark that<sup>55</sup>  $\{\overline{Z_i} \mid 0 \leq i \leq m\} \cup \{O_i \mid i \leq m\}$ , is a partition of  $\Sigma$ . Thus condition (FC2) implies that

$$\mathfrak{F}_0(b\mathcal{K}) = \bigsqcup_{i=0}^m (\mathfrak{F}_0(b\mathcal{K}) \cap \overline{Z_i}) \quad (5.54)$$

Recall also (cf. Eq. (3.1) above) that

$$\mathfrak{F}_0(b\mathcal{K}) = \mathfrak{F}_0(K_1) \sqcup \mathfrak{F}_0(K_1|K_2) \sqcup \mathfrak{F}_0(K_2) \quad (5.55)$$

(Here and in the following  $S \sqcup T$  denotes the disjoint union of any two sets  $S$  and  $T$ ).

In the following let  $B \in \mathcal{B}(b\mathcal{K})$  be of the form

$$B = \frac{1}{k}(\alpha_0 + \sum_i \alpha_i \cdot \star D_i), \quad \text{with } \alpha_0, \dots, \alpha_m \in \Lambda \quad (5.56)$$

and with  $D_1, \dots, D_m$  as above (cf. Eqs. (5.27) and (5.32) in Sec. 5.4.2).

**Lemma 3** *If  $B \in \mathcal{B}(b\mathcal{K})$  is of the form (5.56) then, for each  $i$ , the restriction of  $B : \mathfrak{F}_0(b\mathcal{K}) \rightarrow \mathfrak{t}$  onto  $\mathfrak{F}_0(b\mathcal{K}) \cap \overline{Z_i}$  is constant.*

*Proof.* From Remark 5.8 in Sec. 5.4.3 above it follows<sup>56</sup> that the restriction of  $B : \mathfrak{F}_0(b\mathcal{K}) \rightarrow \mathfrak{t}$  to  $\mathfrak{F}_0(b\mathcal{K}) \cap Z_i$  is constant. The assertion of the lemma now follows from Remark 3.4 ii) in Sec. 3.2.3.

□

<sup>55</sup>here  $\overline{Z_i}$  denotes the closure of  $Z_i$

<sup>56</sup>since  $Z_i$  is a connected component of  $\Sigma \setminus (\bigcup_j (\text{arc}(l_\Sigma^j) \cup \text{arc}(l_\Sigma'^j)))$



**Lemma 4** For every  $B \in \mathcal{B}(b\mathcal{K})$  of the form (5.56) and fulfilling  $\prod_x 1_{\text{treg}}(B(x)) \neq 0$  we have

$$\text{Det}^{\text{disc}}(B) \sim \prod_{i=0}^m \det(1_{\mathfrak{k}} - \exp(\text{ad}(B(Y_i))))_{|\mathfrak{k}}^{\chi(Y_i)/2} \quad (5.57)$$

where  $\chi(Y_i)$  is the Euler characteristic of  $Y_i$  and where we set<sup>57</sup>  $B(Y_i) := B(Z_i) := B(x)$  for any  $x \in Z_i \cap \mathfrak{F}_0(b\mathcal{K})$

*Proof.* From the definition of  $\text{Det}^{\text{disc}}(B)$  in (5.18) and Eqs. (4.4) and (4.9) in Sec. 4 above it follows that

$$\text{Det}^{\text{disc}}(B) \sim \left( \frac{\prod_{x \in \mathfrak{F}_0(K_1)} \det(1_{\mathfrak{k}} - \exp(\text{ad}(B(x))))_{|\mathfrak{k}} \prod_{x \in \mathfrak{F}_0(K_2)} \det(1_{\mathfrak{k}} - \exp(\text{ad}(B(x))))_{|\mathfrak{k}}}{\prod_{x \in \mathfrak{F}_0(K_1|K_2)} \det(1_{\mathfrak{k}} - \exp(\text{ad}(B(x))))_{|\mathfrak{k}}} \right)^{1/2} \quad (5.58)$$

(observe that the expression on the RHS is well-defined since by assumption  $\prod_x 1_{\text{treg}}(B(x)) \neq 0$ , which implies that the denominator is non-zero).

According to Lemma 3 and Eqs. (5.54) and (5.55) it is enough to prove that for each  $i \in \{0, 1, \dots, m\}$  we have

$$\chi(Y_i) = \#(\mathfrak{F}_0(K_1) \cap \overline{Z_i}) - \#(\mathfrak{F}_0(K_1|K_2) \cap \overline{Z_i}) + \#(\mathfrak{F}_0(K_2) \cap \overline{Z_i}) \quad (5.59)$$

Clearly,  $\chi(Y_i) = \chi(\overline{Y_i})$  where  $\overline{Y_i}$  is the closures of  $Y_i$ . Moreover,  $\overline{Y_i}$  is a subcomplex of the CW complex  $K_1 = \mathcal{K} = (\Sigma, \mathcal{C})$  so setting  $\text{Cell}_p(\overline{Y_i}) := \{\sigma \in \text{Cell}_p(K_1) \mid \sigma \subset \overline{Y_i}\}$  where  $\text{Cell}_p(K_1)$  is the set of (open)  $p$ -cells of  $K_1$  we obtain

$$\chi(\overline{Y_i}) = \sum_{p=0}^2 (-1)^p \# \text{Cell}_p(\overline{Y_i}) \stackrel{(*)}{=} \#(\mathfrak{F}_0(K_1) \cap \overline{Y_i}) - \#(\mathfrak{F}_0(K_1|K_2) \cap \overline{Y_i}) + \#(\mathfrak{F}_0(K_2) \cap \overline{Y_i}) \quad (5.60)$$

(step  $(*)$  follows by taking into account the natural 1-1-correspondences  $\text{Cell}_0(K_1) \leftrightarrow \mathfrak{F}_0(K_1)$ ,  $\text{Cell}_1(K_1) \leftrightarrow \mathfrak{F}_0(K_1|K_2)$ , and  $\text{Cell}_2(K_1) \leftrightarrow \mathfrak{F}_0(K_2)$ ).

In order to complete the proof of Lemma 4 it is therefore enough to show that the RHS of Eq. (5.59) and the RHS of Eq. (5.60) coincide.

In order to see this observe that for each  $0 \leq i \leq m$  there is  $J \subset \{0, 1, \dots, m\}$  such that

$$\overline{Y_i} = \overline{Z_i} \sqcup \bigsqcup_{j \in J} (O_j \sqcup \text{arc}(l_{\Sigma}^{'j}))$$

so our claim follows from

$$\begin{aligned} & \#(\mathfrak{F}_0(K_1) \cap \text{arc}(l_{\Sigma}^{'j})) - \#(\mathfrak{F}_0(K_1|K_2) \cap \text{arc}(l_{\Sigma}^{'j})) + \#(\mathfrak{F}_0(K_2) \cap \text{arc}(l_{\Sigma}^{'j})) \\ &= -\#(\mathfrak{F}_0(K_1|K_2) \cap \text{arc}(l_{\Sigma}^{'j})) + \#(\mathfrak{F}_0(K_2) \cap \text{arc}(l_{\Sigma}^{'j})) = 0 \end{aligned} \quad (5.61a)$$

and from

$$\#(\mathfrak{F}_0(K_1) \cap O_j) - \#(\mathfrak{F}_0(K_1|K_2) \cap O_j) + \#(\mathfrak{F}_0(K_2) \cap O_j) = \#\emptyset - \#\emptyset + \#\emptyset = 0 \quad (5.61b)$$

(cf. condition (FC2) and Eq. (5.55)).

□

**Lemma 5** For every  $B \in \mathcal{B}(b\mathcal{K})$  of the form (5.56) we have

$$\prod_{x \in \mathfrak{F}_0(b\mathcal{K})} 1_{\text{treg}}(B(x)) = \prod_{i=0}^m 1_{\text{treg}}(B(Y_i)) \quad (5.62)$$

---

<sup>57</sup>according to Lemma 3 the value of  $B(Y_i) = B(Z_i)$  does not depend on the choice of  $x \in Z_i \subset Y_i$

*Proof.* The assertion follows from Lemma 3, the definition of  $B(Y_i)$  in Lemma 4, and Eq. (5.54) above.  $\square$

Combining Eq. (5.53) with Lemma 4 and Lemma 5 we arrive at

$$\begin{aligned} \text{WLO}_{rig}^{disc}(L) &\sim \sum_{\alpha_0, \alpha_1, \dots, \alpha_m \in \Lambda} 1_{kQ}(\alpha_0) \left( \prod_{i=1}^m m_{\chi_i}(\alpha_i) \right) \\ &\times \left[ \prod_{i=0}^m 1_{\text{treg}}(B(Y_i)) \det(1_{\mathfrak{k}} - \exp(\text{ad}(B(Y_i)))|_{\mathfrak{k}}) \right]^{\chi(Y_i)/2} \\ &\times \prod_{j=1}^m \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) \Big]_{|B=\frac{1}{k}(\alpha_0 + \sum_i \alpha_i \cdot \star D_i)} \end{aligned} \quad (5.63)$$

#### 5.4.6 Step 6: Comparison of $\text{WLO}_{rig}(L)$ with the shadow invariant $|L|$

From the computations in Sec. 5 in [18] it follows that the RHS of Eq. (5.63) above coincides with the shadow invariant  $|L|$  (associated to  $G$  and  $k$ ) up to a multiplicative constant. For the convenience of the reader we will briefly sketch this derivation. In the following we will use the notation of part B of the Appendix.

For  $\alpha_1, \dots, \alpha_m \in \Lambda$  and  $\alpha_0 \in \Lambda \cap kQ$  set

$$B := \frac{1}{k}(\alpha_0 + \sum_i \alpha_i \cdot \star D_i) \quad (5.64)$$

and introduce the function  $\varphi : \{Y_0, Y_1, \dots, Y_m\} \rightarrow \Lambda$  by

$$\varphi(Y) := kB(Y) - \rho \quad (5.65)$$

One can show that then (cf. Sec. 5 in [18])

$$\det(1_{\mathfrak{k}} - \exp(\text{ad}(B(Y)))|_{\mathfrak{k}}) \sim \dim(\varphi(Y))^2 \quad (5.66a)$$

$$\prod_j \exp(\pi i \epsilon_j \langle \alpha_j, B(\sigma_j) + B(\sigma'_j) \rangle) = \prod_Y \exp(\frac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle)^{\text{gleam}(Y)} \quad (5.66b)$$

Let  $P$  be the unique Weyl alcove which is contained in the Weyl chamber  $\mathcal{C}$  fixed in Sec. 2.1 and which has  $0 \in \mathfrak{t}$  on its boundary. Moreover, let  $\Lambda_+^k$  and  $\mathcal{W}_k \cong \mathcal{W}_{\text{aff}}$  be as in part B of the Appendix and let  $\text{col}(L) = (\Lambda_+^k)^{\{Y_0, Y_1, \dots, Y_m\}}$  (the set of “area colorings”). From the relation  $\Lambda_+^k = \Lambda \cap (Pk - \rho)$ , the bijectivity of the map  $\theta : P \times \mathcal{W}_{\text{aff}} \ni (b, \sigma) \mapsto \sigma(b) \in \mathfrak{t}_{\text{reg}}$  and the fact that for a suitable finite subset  $W$  of  $\mathcal{W}_{\text{aff}} (\cong \mathcal{W}_k)$  we have  $\theta(P \times W) = Q \cap \mathfrak{t}_{\text{reg}}$  it follows that there is a natural 1-1-correspondence between the set  $\text{col}(L) \times W \times \mathcal{W}_k^{\{Y_1, \dots, Y_m\}}$  and the set of those  $B$  which are of the form in Eq. (5.64) above (with  $\alpha_0 \in \Lambda \cap kQ$  and  $\alpha_1, \dots, \alpha_m \in \Lambda$ ) and which have the extra property that  $\prod_Y 1_{\text{treg}}(B(Y)) = 1$ .

Using this and Eq. (B.7) below plus a suitable symmetry argument based on the group  $\mathcal{W}_k$  (cf. the proof of Theorem 5.1 in [18]) one then arrives at<sup>58</sup>

$$\begin{aligned} \text{WLO}_{rig}^{disc}(L) &\sim \sum_{\varphi \in \text{col}(L)} \left( \prod_i N_{\gamma(l_i)\varphi(Y_i^+)}^{\varphi(Y_i^-)} \right) \\ &\times \left( \prod_Y \dim(\varphi(Y))^{\chi(Y)} \exp(\frac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle)^{\text{gleam}(Y)} \right) = |L| \end{aligned} \quad (5.67)$$

Recall that  $\sim$  above denotes equality up to a multiplicative constant independent of  $L$ . Thus Eq. (5.67) applied to  $L$  and to the empty link  $\emptyset$  instead of  $L$  implies

$$\frac{\text{WLO}_{rig}^{disc}(L)}{|L|} = \frac{\text{WLO}_{rig}^{disc}(\emptyset)}{|\emptyset|}$$

<sup>58</sup>the multiplicities  $m_{\chi_i}(\alpha_i)$ ,  $i \leq m$ , appearing in Eq. (5.63) lead to the fusion coefficients  $N_{\mu\nu}^\lambda$  appearing in Eq. (5.67) below, cf. the RHS of Eq. (B.7)

Since by definition we have  $\text{WLO}_{rig}(L) = \frac{\text{WLO}_{rig}^{disc}(L)}{\text{WLO}_{rig}^{disc}(\emptyset)}$ , Eq. (5.1) now follows.

## 6 Some comments regarding general links

Let us make some comments regarding the question if it is possible to generalize the computations above to general links. The crucial step will be the evaluation of the integral

$$\int_{\sim} \prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_i^{disc}(\check{A}^\perp, A_c, B; h)) d\nu_B^{disc}(\check{A}^\perp) \quad (6.1)$$

for given  $B$ ,  $A_c$ , and  $h$ , cf. Eq. (5.8) above. For general links this is considerably more difficult than in Step 1 in the proof of Theorem 5.3. The good news is that the evaluation of the integral (6.1) can be reduced to the computation of the “2-clusters”

$$\int_{\sim} \{ \rho_i(\exp(\sum_a T_a Y_k^{i,a})) \otimes \rho_{i'}(\exp(\sum_{a'} T_{a'} Y_{k'}^{i',a'})) \} d\nu_B^{disc} \in \text{End}(V_i) \otimes \text{End}(V_{i'}) \quad (6.2)$$

for the few  $i, i' \leq m$  and  $k, k' \leq n$  for which  $\star l_\Sigma^{i(k)} = l_\Sigma^{i'(k')}$ . Here  $Y_k^{i,a}$  and  $Y_{k'}^{i',a'}$  are as in Eq. (5.9) above<sup>59</sup> and  $V_i, V_{i'}$  are the representation spaces of  $\rho_i$  and  $\rho_{i'}$ , cf. Sec. 5.4.1. The integral in (6.1) above can be expressed by these “2-clusters” by a similar formula as Eq. (6.4) in [25] (cf. also Sec. 5.3 in [28] and [40, 16]).

The explicit formula for  $\text{WLO}_{rig}(L)$  for general<sup>60</sup>  $L$  which one obtains in this way should again be a sum over the set of “area coloring”  $\varphi$ , but this time every summand will contain an extra factor involving a product  $\prod_{x \in V(L)} \dots$ . One could hope that this factor coincides with the factor  $|L|_4^\varphi$  (cf. part B of the Appendix for the notation used here).

In order to evaluate the chances for this being the case we can<sup>61</sup> consider the case of Abelian structure group  $G = U(1)$ . The computations are then analogous to those appearing in the continuum setting in Secs 5.1 and 6.1 in [28] (which led to the correct result). However, in these computations there is one crucial difference in comparison to the computations in the continuum computations: there are several factors of  $1/2$ , coming from the RHS of Eq. (4.3) above, which “spoil” the final result. So ultimately we do *not* recover the (correct) expressions which appeared in the continuum setting. This complication can be resolved<sup>62</sup> by making the transition to the “ $BF_3$ -theory point of view”.

## 7 Outlook: Transition to the “ $BF_3$ -theory point of view”

### 7.1 Motivation

The simplicial program for Abelian CS theory (cf. part F of the Appendix) was completed successfully by D.H. Adams, see [1, 2]. A crucial step in [1, 2] was the transition to the “BF-theory point of view”, which can be divided into two steps, namely “field doubling”<sup>63</sup> followed by a suitable linear change of variables, cf. part C of the Appendix below.

<sup>59</sup>in fact, one can restrict oneself to the special case where  $h = [0]$  in Eq. (5.9)

<sup>60</sup>for general  $L$  and framings the conditions (FC1)-(FC4) and the definition of the open sets  $O_i$  must be generalized in a suitable way. We choose again  $\mathcal{O} = \bigcup_i O_i$ , cf. in Sec. 5.2. The general definition of  $O_i$ ,  $i \leq m$ , is intuitively clear: it will again be the open set “between”  $l_\Sigma^i$  and  $l_\Sigma^{i'}$

<sup>61</sup>studying the case  $G = U(1)$  is instructive even though, strictly speaking, the heuristic equations Eq. (2.8) and Eq. (3.6) were only derived for simply-connected compact groups  $G$  and therefore do not include the case  $G = U(1)$ . Let us mention that in the case  $G = U(1) \times U(1)$  and  $(k_1, k_2) = (k, -k)$  an analogue of Eq. (7.3) below can be derived, cf. Remark 7.1 below.

<sup>62</sup>observe that on the RHS of Eq. (7.14) below most of the  $1/2$ -factors appearing in Eq. (4.3) are absent

<sup>63</sup>which can come in the form of “group doubling” (see Step 1 below) or “base manifold doubling”; observe that the word “doubling” is slightly misleading because it ignores a sign change: we have  $k_2 = -k_1$  where  $k_1, k_2$  are as in the first paragraph of “Step 1” below

Adams's results seem to suggest that – if one wants to have a chance of carrying out the simplicial program successfully also for Non-Abelian CS theory – then a similar strategy will have to be used.

So far we have worked with the original CS point of view because this helped us to reduce the lengths of many formulas considerably and because for Theorem 5.3 (which deals only with a special class of links  $L$ ) the original CS point of view is sufficient. On the other hand, the Abelian “test situation” which we considered at the end of Sec. 6 showed us that we can not expect to obtain correct results within the original CS point of view when dealing with general links. So now is the time to finally focus on the “ $BF_3$ -theory point of view”.

## 7.2 Transition to the “ $BF_3$ -theory point of view”

In the following we will make the transition from non-Abelian CS theory in the torus gauge to the corresponding “ $BF_3$ -theory point of view” at a heuristic level.

### Step 1: “Group doubling”

Let us now consider the version of Eq. (2.8) in the special case where  $G = \tilde{G} \times \tilde{G}$  where  $\tilde{G}$  is a simple, simply-connected compact Lie group and where  $(k_1, k_2)$  fulfills  $k_1 = -k_2$ , cf. Remark 2.7 in Sec. 2.3 of [30]. We set  $k := k_1 = -k_2$ .

For simplicity, let us consider the special case where each of the representations  $\rho_i$  appearing in Eq. (2.8) is of the form  $\rho_i(\tilde{g}_1, \tilde{g}_2) = \tilde{\rho}_i(\tilde{g}_1)$ ,  $\tilde{g}_1, \tilde{g}_2 \in \tilde{G}$ , for a some  $\tilde{G}$ -representation  $\tilde{\rho}_i$ . In this situation we should have<sup>64</sup>

$$\text{WLO}(L) \sim \text{WLO}_{\tilde{G}}(L) \overline{\text{WLO}_{\tilde{G}}(\emptyset)} \sim |L| \cdot |\overline{\emptyset}| \stackrel{(*)}{=} |L| \cdot |\emptyset| \quad (7.1)$$

where  $\text{WLO}_{\tilde{G}}(L)$  on the RHS is defined as  $\text{WLO}(L)$  in Sec. 2.1 for the group  $\tilde{G}$  instead of  $G$  and where  $|\cdot|$  is now the shadow invariant for  $\tilde{G}$  and  $k$ . In step  $(*)$  we used the fact that  $|\emptyset|$  is a real number.

Let us fix a maximal torus  $\tilde{T}$  of  $\tilde{G}$  and take  $T = \tilde{T} \times \tilde{T}$ . Let  $\mathcal{B}$ ,  $\mathcal{A}^\perp$ ,  $\check{\mathcal{A}}^\perp$ ,  $\mathcal{A}_c^\perp$ , and  $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp}$  be defined as in Sec. 2.1 and Sec. 2.2 for the group  $G = \tilde{G} \times \tilde{G}$ .

In the following  $B_1, B_2$  (resp.  $A_1^\perp$  and  $A_2^\perp$ ) will denote the two components of  $B \in C^\infty(\Sigma, \mathfrak{t}) = C^\infty(\Sigma, \mathfrak{t}) \oplus C^\infty(\Sigma, \mathfrak{t})$  (resp.  $A^\perp \in C^\infty(S^1, \mathcal{A}_{\Sigma, \mathfrak{g}}) = C^\infty(S^1, \mathcal{A}_{\Sigma, \mathfrak{g}}) \oplus C^\infty(S^1, \mathcal{A}_{\Sigma, \mathfrak{g}})$ ). Moreover, we make the identification  $[\Sigma, G/T] = [\Sigma, (\tilde{G} \times \tilde{G})/(\tilde{T} \times \tilde{T})] \cong [\Sigma, \tilde{G}/\tilde{T}]^2$ .

### Step 2: Linear change of variable

As we explain in part C of the Appendix, CS theory with group  $G = \tilde{G} \times \tilde{G}$  and  $(k_1, k_2) = (k, -k)$  is equivalent to  $BF_3$ -theory with group  $\tilde{G}$  and “cosmological constant”  $\Lambda$  given by  $\Lambda = \frac{1}{k^2}$ . More precisely, at the heuristic level, these two theories are related by a simple linear change of variables<sup>65</sup>, cf. Eqs. (C.5) or Eqs. (C.10) in part C of the Appendix depending on whether we are dealing with the non-gauge fixed path integral or the path integral in the torus gauge.

In order to simplify the notation a bit (and to avoid the appearance of multiple  $k$ -factors) we will work with the following simplified change of variable  $\check{A}^\perp \rightarrow \tilde{A}^\perp$ ,  $A_c \rightarrow \tilde{A}_c$ ,  $B \rightarrow \tilde{B}$  instead of the one in Eq. (C.10):

$$\tilde{A}^\perp := \left( \frac{\check{A}_1^\perp + \check{A}_2^\perp}{2}, \frac{\check{A}_1^\perp - \check{A}_2^\perp}{2} \right), \quad (7.2a)$$

$$\tilde{A}_c := \left( \frac{(A_c)_1 + (A_c)_2}{2}, \frac{(A_c)_1 - (A_c)_2}{2} \right), \quad (7.2b)$$

$$\tilde{B} := \left( \frac{B_1 + B_2}{2}, \frac{B_1 - B_2}{2} \right) \quad (7.2c)$$

<sup>64</sup>the first “ $\sim$ ” follows from a short heuristic computation

<sup>65</sup>observe that  $\kappa = \frac{1}{k}$  in Eqs. (C.5) and (C.10)

By applying this linear change of variable to the RHS of Eq. (2.8) (in the special case  $G = \tilde{G} \times \tilde{G}$ ,  $(k_1, k_2) = (k, -k)$ ) we arrive at<sup>66</sup>

$$\begin{aligned} \text{WLO}(L) \sim & \sum_{(h_1, h_2) \in [\Sigma, \tilde{G}/\tilde{T}]^2} \int_{\tilde{\mathcal{A}}_c \times \tilde{\mathcal{B}}} 1_{C^\infty(\Sigma, \tilde{\mathfrak{t}}_{reg} \times \tilde{\mathfrak{t}}_{reg})}((\tilde{B}_1 + \tilde{B}_2, \tilde{B}_1 - \tilde{B}_2)) \text{Det}_{FP}((\tilde{B}_1 + \tilde{B}_2, \tilde{B}_1 - \tilde{B}_2)) \\ & \times \left[ \int_{\tilde{\mathcal{A}}^\perp} \prod_i \text{Tr}_{\tilde{\rho}_i}(\text{Hol}_{l_i}(\check{\tilde{A}}_1^\perp + \check{\tilde{A}}_2^\perp + (\tilde{A}_c)_1 + (\tilde{A}_c)_2, \tilde{B}_1 + \tilde{B}_2; h_1)) \exp(i\mathbb{S}(\check{\tilde{A}}^\perp, \tilde{B})) D\check{\tilde{A}}^\perp \right] \\ & \times \exp(2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \text{Tr}(dA_{sg}((h_1, h_2))((\tilde{B}_1 + \tilde{B}_2, \tilde{B}_1 - \tilde{B}_2))) \exp(i\mathbb{S}(\tilde{A}_c, \tilde{B}))(D\tilde{A}_c \otimes D\tilde{B})) \quad (7.3) \end{aligned}$$

where for reasons of notational consistency, we have written  $\check{\tilde{A}}^\perp$  instead of  $\tilde{\mathcal{A}}^\perp$  and  $\tilde{\mathcal{B}}$  and  $\mathcal{B}$  and where we have set

$$\tilde{\mathcal{A}}_c := \mathcal{A}_{\Sigma, \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{t}}}$$

and

$$\begin{aligned} \mathbb{S}(\check{\tilde{A}}^\perp, \tilde{B}) &:= S_{CS}(\check{\tilde{A}}^\perp, B) \\ \mathbb{S}(\tilde{A}_c, \tilde{B}) &:= S_{CS}(A_c, B) \end{aligned}$$

More explicitly, we have<sup>67</sup>

$$\begin{aligned} \mathbb{S}(\check{\tilde{A}}^\perp, \tilde{B}) &= S_{CS}((\check{\tilde{A}}_1^\perp + \check{\tilde{A}}_2^\perp, \check{\tilde{A}}_1^\perp - \check{\tilde{A}}_2^\perp), (\tilde{B}_1 + \tilde{B}_2, \tilde{B}_1 - \tilde{B}_2)) \\ &= S_{CS}(\check{\tilde{A}}_1^\perp + \check{\tilde{A}}_2^\perp, \tilde{B}_1 + \tilde{B}_2) - S_{CS}(\check{\tilde{A}}_1^\perp - \check{\tilde{A}}_2^\perp, \tilde{B}_1 - \tilde{B}_2) \\ &= S_{CS}(\check{\tilde{A}}_1^\perp + \check{\tilde{A}}_2^\perp + (\tilde{B}_1 + \tilde{B}_2)dt) - S_{CS}(\check{\tilde{A}}_1^\perp - \check{\tilde{A}}_2^\perp + (\tilde{B}_1 - \tilde{B}_2)dt) \\ &= \pi k \ll (\check{\tilde{A}}_1^\perp, \check{\tilde{A}}_2^\perp), \left( \begin{array}{cc} \star \text{ad}(\tilde{B}_2) & \star(\frac{\partial}{\partial t} + \text{ad}(\tilde{B}_1)) \\ \star(\frac{\partial}{\partial t} + \text{ad}(\tilde{B}_1)) & \star \text{ad}(\tilde{B}_2) \end{array} \right) \cdot (\check{\tilde{A}}_1^\perp, \check{\tilde{A}}_2^\perp) \gg_{\check{\tilde{A}}^\perp} \quad (7.4) \end{aligned}$$

and

$$\begin{aligned} \mathbb{S}(\tilde{A}_c, \tilde{B}) &= S_{CS}((\tilde{A}_c)_1 + (\tilde{A}_c)_2, (\tilde{A}_c)_1 - (\tilde{A}_c)_2, (\tilde{B}_1 + \tilde{B}_2, \tilde{B}_1 - \tilde{B}_2)) \\ &= \text{ldots} \\ &= -4\pi k \ll ((\tilde{A}_c)_1, (\tilde{A}_c)_2), \left( \begin{array}{cc} 0 & \star d \\ \star d & 0 \end{array} \right) \cdot (\tilde{B}_1, \tilde{B}_2) \gg_{\mathcal{A}_{\Sigma, \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{t}}}} \quad (7.5) \end{aligned}$$

where  $\check{\tilde{A}}^\perp = (\check{\tilde{A}}_1^\perp, \check{\tilde{A}}_2^\perp)$ ,  $\tilde{A}_c = ((\tilde{A}_c)_1, (\tilde{A}_c)_2)$ , and  $\tilde{B} = (\tilde{B}_1, \tilde{B}_2)$ .

**Remark 7.1** Recall that above we have assumed that  $\tilde{G}$  is a simple, simply-connected compact (and therefore non-Abelian) Lie group. For sake of completeness (and in view of the discussion at the end of Sec. 6 above and Remark 7.2 and Remark 7.3 below) let us mention that, in fact, Eq. (7.3) is also valid at a heuristic level if  $\tilde{G}$  is an Abelian compact Lie group. Of course, in this case the RHS of Eq. (7.3) simplifies drastically<sup>68</sup>.

<sup>66</sup>observe also that because of our assumption that  $\rho_i$  has the special form above we have  $\text{Tr}_{\rho_i}(\text{Hol}_{l_i}((\check{\tilde{A}}^\perp, B; (h_1, h_2))) = \text{Tr}_{\tilde{\rho}_i}(\text{Hol}_{l_i}(\check{\tilde{A}}_1^\perp, B_1; h_1))$

<sup>67</sup>the first appearance of  $S_{CS}$  on the RHS of the following equation is a short hand for  $S_{CS}(M, \tilde{G} \times \tilde{G}, (k_1, k_2))$  while the other appearances are a shorthand for  $S_{CS}(M, \tilde{G}, k_1) = S_{CS}(M, \tilde{G}, -k_2)$ .

<sup>68</sup>we then have  $\tilde{T} = \tilde{G}$ ,  $\tilde{\mathfrak{t}}_{reg} = \tilde{\mathfrak{t}}$ ,  $\text{ad}(\tilde{B}_j) = 0$ ,  $A_{sg}(h_1, h_2) = 0$ , and the sum  $\sum_{h_1, h_2}$  is trivial. So we obtain:  
 $\text{WLO}(L) \sim \int [\int \prod_i \text{Tr}_{\tilde{\rho}_i}(\text{Hol}_{l_i}(\check{\tilde{A}}_1^\perp + \check{\tilde{A}}_2^\perp + (\tilde{A}_c)_1 + (\tilde{A}_c)_2, \tilde{B}_1 + \tilde{B}_2; 0)) \exp(i\mathbb{S}(\check{\tilde{A}}^\perp, \tilde{B})) D\check{\tilde{A}}^\perp] \exp(i\mathbb{S}(\tilde{A}_c, \tilde{B}))(D\tilde{A}_c \otimes D\tilde{B})$

**Remark 7.2** It might seem surprising that the second of the aforementioned two steps, i.e. the linear change of variable, really makes an essential difference. Clearly, the original heuristic path integral and the heuristic path integral after the application of the change of variable are equivalent. However, once the problem of discretizing the corresponding path integral is considered the difference really matters. A detailed look at [1, 2] will convince the reader that this is indeed the case at least in the Abelian situation.

That a linear change of variables is useful also for the discretization of non-Abelian CS (with doubled group) is less obvious. Observe, for example, that there is a  $\star$ -operator on the main diagonal of the  $2 \times 2$ -matrix appearing in Eq. (7.4) above. Because of this we cannot hope to be able to find a discretized version of the path integral on the RHS of Eq. (7.3) where each of the two components  $\check{A}_1^\perp$  and  $\check{A}_2^\perp$  “lives” either on  $K_1 \times \mathbb{Z}_N$  or on  $K_2 \times \mathbb{Z}_N$ . Instead, each component  $\check{A}_1^\perp$  and  $\check{A}_2^\perp$  must be implemented in a “mixed” fashion<sup>69</sup>. This is a crucial difference compared to the Abelian situation where it was indeed possible to find a non-mixed discretization for the relevant simplicial fields. This difference is one of the reasons why we decided to postpone the transition to the  $BF_3$ -theory point of view until now.

### 7.3 Discretization of Eq. (7.3)

We will now sketch how – using a suitable discretization of the expression on the RHS of Eq. (7.3) – a rigorous definition of  $WLO(L)$  appearing in Eq. (7.3) can be obtained. (In part E of the Appendix we will sketch two alternative ways of discretizing the RHS of Eq. (7.3)).

In order to simplify the notation we will now drop all the  $\sim$ -signs appearing in the previous subsection, for example will write  $G$  instead of  $\tilde{G}$  and  $\mathcal{B}$  instead of  $\tilde{\mathcal{B}}$  and so on. Clearly, we then have

$$\mathcal{B} = C^\infty(\Sigma, \mathfrak{t} \oplus \mathfrak{t}) \quad (7.6a)$$

$$\check{\mathcal{A}}^\perp = \{A^\perp \in C^\infty(S^1, \mathcal{A}_{\Sigma, \mathfrak{g} \oplus \mathfrak{g}}) \mid \int A^\perp(t) dt \in \mathcal{A}_{\Sigma, \mathfrak{t} \oplus \mathfrak{t}}\} \quad (7.6b)$$

$$\mathcal{A}_c = \Omega^1(\Sigma, \mathfrak{t} \oplus \mathfrak{t}) \quad (7.6c)$$

Let us now introduce the space

$$\mathcal{E} := \Omega^2(\Sigma, \mathfrak{t} \oplus \mathfrak{t}) \quad (7.6d)$$

and rewrite Eq. (7.3) in a similar way as we rewrote Eq. (2.8) in Sec. 3.2.1. By doing so (and using the notation  $h_+$  instead of  $h_1$  and  $h_-$  instead of  $h_2$ ) we obtain

$$\begin{aligned} WLO(L) \sim & \sum_{(h_+, h_-) \in [\Sigma, G/T]^2} \int_{\mathcal{A}_c \times \mathcal{E}} \left\{ 1_{C^\infty(\Sigma, \mathfrak{t}_{reg} \times \mathfrak{t}_{reg})}(B_+, B_-) \text{Det}_{FP}(B_+, B_-) \right. \\ & \times \left[ \int_{\check{\mathcal{A}}^\perp} \prod_i \text{Tr}_{\rho_i}(\text{Hol}_i(A_1^\perp + A_2^\perp, B_+; h_+)) \exp(i\mathbb{S}(\check{A}^\perp, B)) D\check{A}^\perp \right] \\ & \times \exp(2\pi i k \int_{\Sigma \setminus \{\sigma_0\}} \text{Tr}(dA_{sg}((h_+, h_-)) \cdot (B_+, B_-))) \Big\}_{|B:=\star E} \exp(i\mathbb{S}(A_c, E))(DA_c \otimes DE) \quad (7.7) \end{aligned}$$

with

$$B_\pm := B_1 \pm B_2 \quad \text{for } B = (B_1, B_2) \in \mathcal{B}$$

and where for each  $l \in \{l_1, l_2, \dots, l_m\}$  we have set<sup>70</sup> (cf. Eqs. (2.5) and (2.7) in Sec. 2.1 above)

$$\begin{aligned} \text{Hol}_l(A_1^\perp + A_2^\perp, B_+; h_+) &:= \text{Hol}_l(\check{A}_1^\perp + \check{A}_2^\perp + (A_c)_1 + (A_c)_2, B_+; h_+) \\ &= \lim_{n \rightarrow \infty} \prod_{k=1}^n \exp\left(\frac{1}{n}(\check{A}_1^\perp + \check{A}_2^\perp + (A_c)_1 + (A_c)_2 + A_{sg}(h_+) + B_1 dt + B_2 dt)(l'(t))\right) \Big|_{t=k/n} \quad (7.8) \end{aligned}$$

<sup>69</sup>in a similar way as the field  $\check{A}^\perp$  was implemented in Approach I-II of the present paper and [30]

<sup>70</sup>clearly, the notation  $\text{Hol}_l(A_1^\perp + A_2^\perp, B_+; h_+)$  is a bit sloppy. If one wants to be strict one should use the notation  $\text{Hol}_l(\check{A}_1^\perp, \check{A}_2^\perp, (A_c)_1, (A_c)_2, B_+; h_+)$  instead

and where we have set (cf. Eq. (7.4) and Eq. (7.5) above)

$$\mathbb{S}(\check{A}^\perp, B) := \pi k \ll (\check{A}_1^\perp, \check{A}_2^\perp), \star \begin{pmatrix} \text{ad}(B_2) & \frac{\partial}{\partial t} + \text{ad}(B_1) \\ \frac{\partial}{\partial t} + \text{ad}(B_1) & \text{ad}(B_2) \end{pmatrix} \cdot (\check{A}_1^\perp, \check{A}_2^\perp) \gg_{\check{\mathcal{A}}^\perp} \quad (7.9)$$

$$\mathbb{S}(A_c, E) := -4\pi k \ll ((A_c)_1, (A_c)_2), \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \cdot (E_1, E_2) \gg_{\mathcal{A}_{\Sigma, \mathfrak{t} \oplus \mathfrak{t}}} \quad (7.10)$$

where  $\star$  is the obvious linear isomorphism  $\star : \check{\mathcal{A}}^\perp \rightarrow \check{\mathcal{A}}^\perp$ .

We will now discretize the RHS of Eq. (7.7). (In part E of the Appendix we will sketch two alternative ways of discretizing the RHS of Eq. (7.7)). In order to do so we introduce spaces

$$\mathcal{B}(b\mathcal{K}) := C^0(b\mathcal{K}, \mathfrak{t} \oplus \mathfrak{t}) \quad (7.11a)$$

$$\check{\mathcal{A}}^\perp(K) := \{A^\perp \in \text{Map}(\mathbb{Z}_N, \mathcal{A}_{\Sigma, \mathfrak{g} \oplus \mathfrak{g}}(K)) \mid \sum_{t \in \mathbb{Z}_N} A^\perp(t) \in \mathcal{A}_{\Sigma, \mathfrak{t} \oplus \mathfrak{t}}(K)\} \quad (7.11b)$$

$$\mathcal{A}_c(b\mathcal{K}) := C^1(b\mathcal{K}, \mathfrak{t} \oplus \mathfrak{t}) \quad (7.11c)$$

$$\mathcal{E}(b\mathcal{K}) := C^2(b\mathcal{K}, \mathfrak{t} \oplus \mathfrak{t}) \quad (7.11d)$$

As the discrete analogues of Eq. (7.9) and Eq. (7.10) above we take

$$\mathbb{S}^{disc}(\check{A}^\perp, B) := \pi k \ll (\check{A}_1^\perp, \check{A}_2^\perp), \star_K R^{(N)}(B) \cdot (\check{A}_1^\perp, \check{A}_2^\perp) \gg_{\check{\mathcal{A}}^\perp(K)} \quad (7.12)$$

$$\mathbb{S}^{disc}(A_c, E) := -4\pi k \ll ((A_c)_1, (A_c)_2), \begin{pmatrix} 0 & \delta_{b\mathcal{K}} \\ \delta_{b\mathcal{K}} & 0 \end{pmatrix} \cdot (E_1, E_2) \gg_{\mathcal{A}_c(b\mathcal{K})} \quad (7.13)$$

where the scalar products  $\ll \cdot, \cdot \gg_{\check{\mathcal{A}}^\perp(K)}$  and  $\ll \cdot, \cdot \gg_{\mathcal{A}_c(b\mathcal{K})}$  are defined in an analogous way as in Sec. 2.3 and Sec. 3.1 above<sup>71</sup> and where for  $B = (B_1, B_2) \in C^0(b\mathcal{K}, \mathfrak{t} \oplus \mathfrak{t})$  we have set<sup>72</sup>

$$R^{(N)}(B) := \begin{pmatrix} \frac{L^{(N)}(B_+) - L^{(N)}(B_-)}{2} & \frac{L^{(N)}(B_+) + L^{(N)}(B_-)}{2} \\ \frac{L^{(N)}(B_+) + L^{(N)}(B_-)}{2} & \frac{L^{(N)}(B_+) - L^{(N)}(B_-)}{2} \end{pmatrix}$$

with  $B_\pm := B_1 \pm B_2$  and  $L^{(N)}(B_0)$  for  $B_0 \in C^0(b\mathcal{K}, \mathfrak{t})$  as in Eq. (3.25) in Sec. 3.3.4 above.

Above the operator  $\star_K : \check{\mathcal{A}}^\perp(K) \rightarrow \check{\mathcal{A}}^\perp(K)$  is defined in an analogous way as the operator  $\star_K$  in the paragraph of Remark 2.4. Moreover, we will again have to introduce an operator  $\star_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{B}(b\mathcal{K})$  in a similar way as in Sec. 3.2.3 using again the set  $\mathcal{O} := \bigcup_i O_i$  where each  $O_i$  is again given as in Sec. 5.2.

Like in Approach I and II above we introduce for each loop  $l \in \{l_1, \dots, l_m\}$  in  $K_1 \times \mathbb{Z}_N$  a framing  $l'$  in  $K_2 \times \mathbb{Z}_N$  and we define in similar<sup>73</sup> way as in Eq. (4.3) above the following framed discrete analogue of the continuum expression  $\text{Hol}_l(A_1^\perp + A_2^\perp, B_+; \mathfrak{h}_+)$  above<sup>74</sup>

$$\begin{aligned} & \text{Hol}_l^{disc}(A_1^\perp + A_2^\perp, B_+; \mathfrak{h}_+) \\ & := \prod_{k=1}^n \exp \left( (\check{A}_1^\perp(\bullet l_{S^1}^{(k)}))(l_\Sigma^{(k)}) + (\check{A}_2^\perp(\bullet l_{S^1}^{(k)}))(l_\Sigma^{(k)}) + (A_c)_1(l_\Sigma^{(k)}) + (A_c)_2(l_\Sigma^{(k)}) \right. \\ & \quad \left. + \frac{1}{2} \left( \int_{l_\Sigma^{(k)}} A_{\text{sg}}(\mathfrak{h}_+) + \int_{l_\Sigma'^{(k)}} A_{\text{sg}}(\mathfrak{h}_+) \right) + B_1(\bullet l_\Sigma^{(k)}) \cdot \frac{1}{N} \text{sgn}(l_{S^1}^{(k)}) + B_2(\bullet l_\Sigma'^{(k)}) \cdot \frac{1}{N} \text{sgn}(l_{S^1}^{(k)}) \right) \end{aligned} \quad (7.14)$$

<sup>71</sup>in particular,  $\ll \cdot, \cdot \gg_{\check{\mathcal{A}}^\perp(K)}$  will be the restriction to  $\check{\mathcal{A}}^\perp(K)$  of the scalar product given by Eq. (2.16) above

<sup>72</sup>Recall that  $L^{(N)}(B_0)$  is a discrete “approximation” of  $\partial_t + \text{ad}(B_0)$  so  $\frac{1}{2}(L^{(N)}(B_+) + L^{(N)}(B_-))$  is a discrete analogue of  $\frac{1}{2}(\partial_t + \text{ad}(B_+) + \partial_t + \text{ad}(B_-)) = \frac{1}{2}(\partial_t + \text{ad}(B_1 + B_2) + \partial_t + \text{ad}(B_1 - B_2)) = \partial_t + \text{ad}(B_1)$ ; similarly  $\frac{1}{2}(L^{(N)}(B_+) - L^{(N)}(B_-))$  is a discrete approximation of  $\frac{1}{2}(\partial_t + \text{ad}(B_1 + B_2) - (\partial_t + \text{ad}(B_1 - B_2))) = \text{ad}(B_2)$

<sup>73</sup>in order to see that the expression (7.14) is a discrete analogue to the continuum expression (7.8), recall the comments and relations appearing in Sec. 5.3 in [30]. In particular, take into account the (continuum) relations  $\check{A}_j^\perp(l'(t)) = \check{A}_j^\perp(l_{S^1}(t))(l_\Sigma'(t))$ ,  $(A_c)_j(l'(t)) = (A_c)_j(l_\Sigma'(t))$ , and  $(B_j dt)(l'(t)) = B_j(l_\Sigma(t)) \cdot dt(l_{S^1}'(t))$  and the “replacements” appearing in the paragraphs after Eq. (5.13) in [30]

<sup>74</sup>the notation  $\text{Hol}_l^{disc}(A_1^\perp + A_2^\perp, B_+; \mathfrak{h}_+)$  is again a bit sloppy. If one wants to be strict one should use a notation like  $\text{Hol}_l^{disc}(\check{A}_1^\perp, \check{A}_2^\perp, (A_c)_1, (A_c)_2, B_+; \mathfrak{h}_+)$



- Remark 7.3** i) The term  $\frac{1}{2}(\int_{l_{\Sigma}^{(k)}} A_{\text{sg}}(h_+) + \int_{l_{\Sigma}'^{(k)}} A_{\text{sg}}(h_+))$  in Eq. (7.14) above looks a bit strange. If one can justify the simplified version Eq. (2.12) of Eq. (2.8) mentioned in Remark 2.2 above then one can work with analogous simplifications in some of the equations and definitions of the present section. In the simplified version of  $\text{Hol}_l^{\text{disc}}(A_1^{\perp} + A_2^{\perp}, B_+; h_+)$  in (7.14) the “strange” term mentioned before will not appear anymore.
- ii) In the special case where  $G$  is Abelian (cf. Remark 7.1 above)  $\text{Hol}_l^{\text{disc}}(A_1^{\perp} + A_2^{\perp}, B_+; h_+)$  does not depend on  $h_+$  so we can use the notation  $\text{Hol}_l^{\text{disc}}(A_1^{\perp} + A_2^{\perp}, B_+)$  instead. Moreover, there are then additional simplifications. For example, in the special case where the (complex) representation  $\rho$  of  $G$  is irreducible (and therefore 1-dimensional) we have

$$\begin{aligned} \text{Tr}_{\rho}(\text{Hol}_l^{\text{disc}}(A_1^{\perp} + A_2^{\perp}, B_+)) &= \rho\left(\prod_{k=1}^n \exp\left((\check{A}_1^{\perp}(\bullet l_{\Sigma^1}^{(k)}))(l_{\Sigma}^{(k)}) + (A_c)_1(l_{\Sigma}^{(k)}) + B_1(\bullet l_{\Sigma}^{(k)})\right)\right) \\ &\times \rho\left(\prod_{k=1}^n \exp\left((\check{A}_2^{\perp}(\bullet l_{\Sigma^1}'^{(k)}))(l_{\Sigma}'^{(k)}) + (A_c)_2(l_{\Sigma}'^{(k)}) + B_2(\bullet l_{\Sigma}'^{(k)}) \cdot \frac{1}{N} \text{sgn}(l_{\Sigma^1}'^{(k)})\right)\right) \end{aligned}$$

We can work with a generalization of the last expression where the second “ $\rho$ ” appearing above is replaced by another finite-dimensional representation  $\rho'$  of  $G$ . By doing so we obtain the torus gauge “analogues” of the results in [1, 2].

The remaining steps for discretizing the RHS of Eq. (7.7) can be carried out in a straightforward way (in complete analogy with the procedure in Secs 4.4–4.8). When then arrive at a rigorous version  $\text{WLO}_{\text{rig}}^{\text{disc}}(L)$  of  $\text{WLO}(L)$  and its normalization

$$\text{WLO}_{\text{rig}}(L) := \frac{\text{WLO}_{\text{rig}}^{\text{disc}}(L)}{\text{WLO}_{\text{rig}}^{\text{disc}}(\emptyset)} \quad (7.15)$$

In view of the heuristic formula Eq. (7.1) and the normalization ansatz in Eq. (7.15) we expect that

$$\text{WLO}_{\text{rig}}(L) = \frac{|L||\emptyset|}{|\emptyset||\emptyset|} = \frac{|L|}{|\emptyset|}$$

where  $|\cdot|$  is the shadow invariant for<sup>75</sup>  $G$  and  $k$ . And this is indeed the case. We have the following BF-theoretic analogue of Theorem 5.3 above:

**Theorem 7.4** *Assume that the link  $L = (l_1, l_2, \dots, l_m)$  fixed above fulfills conditions (NCP) and (NH) above and that the framings  $(l_1', l_2', \dots, l_m')$  fulfill conditions (FC1)–(FC4) above. Assume also that  $k \geq c_g$ . Then  $\text{WLO}_{\text{rig}}(L)$  given by Eq. (7.15) is well-defined and we have*

$$\text{WLO}_{\text{rig}}(L) = \frac{|L|}{|\emptyset|} \quad (7.16)$$

It remains to be seen if Theorem 7.4 can be generalized to the case of arbitrary links  $L$  in  $M = \Sigma \times S^1$ , cf. [31] for ongoing work in this direction<sup>76</sup>. If this turns out to be the case, then this should lead to some progress in the simplicial program for non-Abelian  $CS/BF_3$ -theory, cf. Appendix F below.

*Acknowledgements:* I want to thank the anonymous referee of my paper [28] whose comments motivated me to look for an alternative approach for making sense of the RHS of Eq. (2.8), which is less technical than the continuum approach in [26, 28, 29]. This eventually led to [30] and the present paper.

<sup>75</sup>recall that we write the group  $\tilde{G}$  now simply as  $G$ , i.e. without the  $\sim$ .

<sup>76</sup>in [31] we are in fact working inside the framework of part E of the Appendix

Moreover, I would like to thank Laurent Freidel for pointing out to me the widespread confusion about the “shift in  $k$ ”-issue mentioned in Remark B.2 below. I am also grateful to Jean-Claude Zambrini for several comments which led to improvements in the presentation of [30] and the present paper.

Finally, it is a great pleasure for me to thank Benjamin Himpel for many useful and important comments and suggestions which not only had a major impact on the presentation and overall structure of [30] and the present paper but also inspired me to reconsider the issue of discretizing the operator  $\frac{\partial}{\partial t} + \text{ad}(B)$ , appearing in Eq. (2.10) above. (This later led to Sec. 3.3 above).

## A Appendix: Lie theoretic notation II

The following two lists extend the two lists in Appendix A in [30].

### A.1 List of notation in the general case

Recall that in Sec. 2.1 we fixed a simply-connected compact Lie group  $G$  (with Lie algebra  $\mathfrak{g}$ ), a maximal  $T$  of  $G$  (with Lie algebra  $\mathfrak{t}$ ), and a Weyl chamber  $\mathcal{C} \subset \mathfrak{t}$ .

Apart from the notation given in Appendix A of [30] we also use the following Lie theoretic notation in the present paper:

- $\langle \cdot, \cdot \rangle$ : the unique Ad-invariant scalar product on  $\mathfrak{g}$  such that<sup>77</sup>  $\langle \check{\alpha}, \check{\alpha} \rangle = 2$  holds for every short real coroot  $\check{\alpha}$  associated to  $(\mathfrak{g}, \mathfrak{t})$ . Using  $\langle \cdot, \cdot \rangle$  we now make the identification  $\mathfrak{t} \cong \mathfrak{t}^*$ .
- $\mathcal{R}_{\mathbb{C}}$ : the set of *complex* roots  $\mathfrak{t} \rightarrow \mathbb{C}$  associated to  $(\mathfrak{g}, \mathfrak{t})$
- $\mathcal{R} \subset \mathfrak{t}^*$ : the set  $\{\frac{1}{2\pi i}\alpha_c \mid \alpha_c \in \mathcal{R}_{\mathbb{C}}\}$  of *real* roots associated to  $(\mathfrak{g}, \mathfrak{t})$
- $\mathcal{R}_+ \subset \mathcal{R}$ : the set of positive (real) roots corresponding to  $\mathcal{C}$
- $\Gamma \subset \mathfrak{t}$ : the lattice generated by the set of real coroots associated to  $(\mathfrak{g}, \mathfrak{t})$ , i.e. by the set  $\{\check{\alpha} \mid \alpha \in \mathcal{R}\}$  where  $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{t}^* \cong \mathfrak{t}$  is the coroot associated to the root  $\alpha \in \mathcal{R}$ .
- $I \subset \mathfrak{t}$ : the kernel of  $\exp|_{\mathfrak{t}} : \mathfrak{t} \rightarrow T$ . From the assumption that  $G$  is simply-connected it follows that  $I = \Gamma$ .
- $\Lambda \subset \mathfrak{t}^*(\cong \mathfrak{t})$ : the *real* weight lattice associated to  $(\mathfrak{g}, \mathfrak{t})$ , i.e.  $\Lambda$  is the lattice which is dual to  $\Gamma$ .
- $\Lambda_+ \subset \Lambda$ : the set of dominant weights corresponding to  $\mathcal{C}$ , i.e.  $\Lambda_+ := \bar{\mathcal{C}} \cap \Lambda$
- $\rho$ : half sum of positive roots (“Weyl vector”)
- $\theta$ : unique long root in the Weyl chamber  $\mathcal{C}$ .
- $c_{\mathfrak{g}} = 1 + \langle \theta, \rho \rangle$ : the dual Coxeter number of  $\mathfrak{g}$ .
- $P \subset \mathfrak{t}$ : a fixed Weyl alcove
- $Q \subset \mathfrak{t}$ : a subset of  $\mathfrak{t}$  of the form  $Q = \{\sum_i \lambda_i e_i \mid 0 < \lambda_i < 1 \forall i \leq r\}$  where  $(e_i)_{i \leq \dim(\mathfrak{t})}$  is a fixed basis of  $\Gamma = I$ .
- $\mathcal{W} \subset \text{GL}(\mathfrak{t})$ : the Weyl group of the pair  $(\mathfrak{g}, \mathfrak{t})$

<sup>77</sup>which is equivalent to the condition  $\langle \alpha, \alpha \rangle_* = 2$  for every *long* real root  $\alpha \in \mathfrak{t}^*$  where  $\langle \cdot, \cdot \rangle_*$  is the scalar product on  $\mathfrak{t}^*$  induced by  $\langle \cdot, \cdot \rangle$

- $\mathcal{W}_{\text{aff}} \subset \text{Aff}(\mathfrak{t})$ : the “affine Weyl group of  $(\mathfrak{g}, \mathfrak{t})$ ”, i.e. the subgroup of  $\text{Aff}(\mathfrak{t})$  generated by  $\mathcal{W}$  and the set of translations  $\{\tau_x \mid x \in \Gamma\}$  where  $\tau_x : \mathfrak{t} \ni b \mapsto b + x \in \mathfrak{t}$ .
- $\mathcal{W}_k \subset \text{Aff}(\mathfrak{t})$ ,  $k \in \mathbb{N}$ : the subgroup of  $\text{Aff}(\mathfrak{t})$  given by  $\{\psi_k \circ \sigma \circ \psi_k^{-1} \mid \sigma \in \mathcal{W}_{\text{aff}}\}$  where  $\psi_k : \mathfrak{t} \ni b \mapsto b \cdot k - \rho \in \mathfrak{t}$  (the “quantum Weyl group corresponding to the level  $l := k - c_{\mathfrak{g}}$ ”)
- $\Lambda_+^k \subset \Lambda$ ,  $k \in \mathbb{N}$ : the subset of  $\Lambda_+$  given by  $\Lambda_+^k := \{\lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_{\mathfrak{g}}\}$  (the “set of dominant weights which are integrable at level  $l := k - c_{\mathfrak{g}}$ ”).

In the main text, the number  $k \in \mathbb{N}$  appearing above will be the integer  $k$  fixed in Sec. 2.1 (which later is assumed to fulfill  $k \geq c_{\mathfrak{g}}$ ).

## A.2 List of notation in the special case $G = SU(2)$

Let us now consider the special group  $G = SU(2)$  with the standard maximal torus  $T = \{\exp(\theta\tau) \mid \theta \in \mathbb{R}\}$  where

$$\tau := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Then  $\mathfrak{g} = \mathfrak{su}(2)$  and  $\mathfrak{t} = \mathbb{R} \cdot \tau$ . There are two Weyl chambers, namely  $\mathcal{C}_+$  and  $\mathcal{C}_-$  where  $\mathcal{C}_{\pm} := \pm[0, \infty)\tau$ . Let us fix  $\mathcal{C} := \mathcal{C}_+$  in the following.

- $\langle \cdot, \cdot \rangle$  is the scalar product on  $\mathfrak{g}$  given by<sup>78</sup>

$$\langle A, B \rangle = -\frac{1}{4\pi^2} \text{Tr}_{\text{Mat}(2, \mathbb{C})}(AB) \quad \text{for all } A, B \in \mathfrak{g} \subset \text{Mat}(2, \mathbb{C})$$

- $\mathcal{R}_{\mathbb{C}} = \{\alpha_c, -\alpha_c\}$  where  $\alpha_c : \mathfrak{t} \rightarrow \mathbb{C}$  is given by  $\alpha_c(\tau) = 2i$
- $\mathcal{R} = \{\alpha, -\alpha\}$  where  $\alpha := \frac{1}{2\pi i} \alpha_c$ . A short computation shows that  $\alpha = 2\pi\tau \in \mathfrak{t}$  (recall that we made the identification  $\mathfrak{t} \cong \mathfrak{t}^*$ ).
- $\mathcal{R}_+ = \{\alpha\}$
- $I = \Gamma = \mathbb{Z} \cdot \check{\alpha}$  where  $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle} = \alpha$
- $\Lambda = \mathbb{Z} \cdot \frac{\alpha}{2}$
- $\Lambda_+ = \mathbb{N}_0 \cdot \frac{\alpha}{2}$
- $\rho = \frac{\alpha}{2}$
- $\theta = \alpha$
- $c_{\mathfrak{g}} = 2$
- possible choices for  $P$  and  $Q$  are  $P = (0, \frac{1}{2})\alpha$  and  $Q = (0, 1)\alpha$
- $\mathcal{W} = \{1, \sigma\}$  where  $1 = \text{id}_{\mathfrak{t}}$  and  $\sigma(b) = -b$  for  $b \in \mathfrak{t}$ ; using this and the explicit description of  $I = \Gamma$  above one easily obtains an explicit description of  $\mathcal{W}_{\text{aff}}$  and  $\mathcal{W}_k$
- $\Lambda_+^k = \{0, \frac{1}{2}\alpha, \dots, \frac{k-2}{2}\alpha\}$  for  $k \in \mathbb{N}$

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<sup>78</sup>in view of the formula  $\check{\alpha} = \alpha = 2\pi\tau$  below we see that  $\langle \cdot, \cdot \rangle$  indeed fulfills the normalization condition  $\langle \check{\alpha}, \check{\alpha} \rangle = 2$

## B Appendix: Turaev's shadow invariant

Let us briefly recall the definition of Turaev's shadow invariant in the situation relevant for us, i.e. for base manifolds  $M$  of the form  $M = \Sigma \times S^1$  where  $\Sigma$  is an oriented surface.

Let  $L = (l_1, l_2, \dots, l_m)$ ,  $m \in \mathbb{N}$ , be a link in  $M = \Sigma \times S^1$  and let  $V(L)$  denote the set of points  $p \in \Sigma$  where the loops  $l_\Sigma^i$ ,  $i \leq m$ , cross themselves or each other (the “crossing points”) and  $E(L)$  the set of curves in  $\Sigma$  into which the loops  $l_\Sigma^1, l_\Sigma^2, \dots, l_\Sigma^m$  are decomposed when being “cut” in the points of  $V(L)$ . We assume that there are only finitely many connected components  $Y_0, Y_1, Y_2, \dots, Y_{m'}, m' \in \mathbb{N}$  (“faces”) of  $\Sigma \setminus (\bigcup_i \text{arc}(l_\Sigma^i))$  and set

$$F(L) := \{Y_0, Y_1, Y_2, \dots, Y_{m'}\}.$$

As explained in [49] one can associate in a natural way a number  $\text{gleam}(Y) \in \mathbb{Z}$ , called “gleam” of  $Y$ , to each face  $Y \in F(L)$ . In the special case which we are considering in Sec. 5, i.e. where the two assumptions (NCP) and (NH) are fulfilled, we have the explicit formula

$$\text{gleam}(Y) = \sum_{i \text{ with } \text{arc}(l_\Sigma^i) \subset \partial Y} \text{wind}(l_{S^1}^i) \cdot \text{sgn}(Y; l_\Sigma^i) \quad (\text{B.1})$$

where  $\text{wind}(l_{S^1}^i)$  is the winding number of the loop  $l_{S^1}^i$  and where  $\text{sgn}(Y; l_\Sigma^i)$  is given by

$$\text{sgn}(Y; l_\Sigma^i) := \begin{cases} 1 & \text{if } Y \subset R_i^+ \\ -1 & \text{if } Y \subset R_i^- \end{cases} \quad (\text{B.2})$$

Here  $R_i^+$  (resp.  $R_i^-$ ) is the unique connected component  $R$  of  $\Sigma \setminus \text{arc}(l_\Sigma^i)$  such that  $l_\Sigma^i$  runs around  $R$  in the “positive” (resp. “negative”) direction.

Let  $G$  be a simply-connected and simple compact Lie group with maximal torus  $T$ . In the following we will use the notation from part A of the Appendix. In particular, we have

$$\Lambda_+^k = \{\lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_{\mathfrak{g}}\} \quad (\text{B.3})$$

where  $k \in \mathbb{N}$  is as in Sec. 2 above and where  $c_{\mathfrak{g}} = 1 + \langle \theta, \rho \rangle$  is the dual Coxeter number of  $\mathfrak{g}$ .

**Remark B.1** Observe that for  $k < c_{\mathfrak{g}}$  the set  $\Lambda_+^k$  is empty so  $|L|$  as defined in Eq. (B.4) below will then vanish. If  $k = c_{\mathfrak{g}}$  then  $|L|$  will not vanish but will still be rather trivial. Not surprisingly, the definition of  $|L|$  in the literature often excludes the situation  $k \leq c_{\mathfrak{g}}$  so our definition of  $|L|$  in Eq. (B.4) below is more general than usually.

Assume that each loop  $l_i$  in the link  $L$  is equipped with a “color”  $\rho_i$ , i.e. a finite-dimensional complex representation of  $G$ . By  $\gamma_i \in \Lambda_+$  we denote the highest weight of  $\rho_i$  and set  $\gamma(e) := \gamma_i$  for each  $e \in E(L)$  where  $i \leq n$  denotes the unique index such that  $\text{arc}(e) \subset \text{arc}(l_i)$ . Finally, let  $\text{col}(L)$  be the set of all mappings  $\varphi : \{Y_0, Y_1, Y_2, \dots, Y_{m'}\} \rightarrow \Lambda_+^k$  (“area colorings”).

We can now define the “shadow invariant”  $|L|$  of the (colored, canonically framed<sup>79</sup>) link  $L$  associated to the pair<sup>80</sup>  $(G, k)$  by

$$|L| := \sum_{\varphi \in \text{col}(L)} |L|_1^\varphi |L|_2^\varphi |L|_3^\varphi |L|_4^\varphi \quad (\text{B.4})$$

<sup>79</sup>by “canonically framed” we mean that each of the loops  $l_i$ ,  $i \leq m$  is equipped with the “vertical” framing, cf. Sec. 4c) in [49]. For links fulfilling (NCP) and (NH) vertical framing is equivalent to what we call “horizontal” framing

<sup>80</sup>Observe that in [48] the shadow invariant is defined for pairs  $(\mathfrak{g}, k)$  where  $k \in \mathbb{N}$  and where  $\mathfrak{g}$  is a simple complex Lie algebra. Since there is a natural 1-1-correspondence between simple complex Lie algebras on the one hand and simply-connected and simple compact Lie groups on the other hand our convention is equivalent to the one in [48]

with

$$|L|_1^\varphi = \prod_{Y \in F(L)} \dim(\varphi(Y))^{\chi(Y)} \quad (\text{B.5a})$$

$$|L|_2^\varphi = \prod_{Y \in F(L)} \exp\left(\frac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle\right)^{\text{gleam}(Y)} \quad (\text{B.5b})$$

$$|L|_3^\varphi = \prod_{e \in E(L)} N_{\gamma(e)\varphi(Y_e^+)}^{\varphi(Y_e^-)} \quad (\text{B.5c})$$

$$|L|_4^\varphi = \prod_{x \in V(L)} T(x, \varphi) \quad (\text{B.5d})$$

Here  $Y_e^+$  (resp.  $Y_e^-$ ) denotes the unique face  $Y$  such that  $\text{arc}(e) \subset \partial Y$  and, additionally, the orientation on  $\text{arc}(e)$  described above coincides with (resp. is opposite to) the orientation which is obtained by restricting the orientation on  $\partial Y$  to  $e$ . Moreover, we have set <sup>81</sup> (for  $\lambda, \mu, \nu \in \Lambda_+^k$ )

$$\dim(\lambda) := \prod_{\alpha \in \mathcal{R}_+} \frac{\sin \frac{\pi \langle \lambda + \rho, \alpha \rangle}{k}}{\sin \frac{\pi \langle \rho, \alpha \rangle}{k}} \quad (\text{B.6})$$

$$N_{\mu\nu}^\lambda := \sum_{\tau \in \mathcal{W}_k} \text{sgn}(\tau) m_\mu(\nu - \tau(\lambda)) \quad (\text{B.7})$$

where  $m_\mu(\beta)$  is the multiplicity of the weight  $\beta$  in the unique (up to equivalence) irreducible representation  $\rho_\mu$  with highest weight  $\mu$  and  $\mathcal{W}_k$  is as in part A of the Appendix.

The explicit expression for  $T(x, \varphi)$  appearing in the formula for  $|L|_4^\varphi$  involves the so-called “quantum 6j-symbols” (cf. Chap. X, Sec. 1.2 in [48]) associated to  $U_q(\mathfrak{g}_\mathbb{C})$  where  $q$  is the root of unity<sup>82</sup>

$$q := \exp\left(\frac{2\pi i}{k}\right) \quad (\text{B.8})$$

We omit the explicit formula for  $T(x, \varphi)$  since it is irrelevant for the present paper: for links  $L$  fulfilling assumption (NCP) of Sec. 5 the set  $V(L)$  is empty and Eq. (B.4) then reduces to

$$|L| = \sum_{\varphi \in \text{col}(L)} \left( \prod_i N_{\gamma(l_i)\varphi(Y_i^+)}^{\varphi(Y_i^-)} \right) \left( \prod_Y \dim(\varphi(Y))^{\chi(Y)} \exp\left(\frac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle\right)^{\text{gleam}(Y)} \right) \quad (\text{B.9})$$

where we have set  $Y_i^\pm := Y_{l_i^\pm}^\pm$ .

**Remark B.2** In the literature on Chern-Simons theory there seems to persist some confusion regarding the precise relationship between the values of the (heuristic) Wilson loop observables in Chern-Simons theory studied by Witten in [50] and the various rigorous versions of Witten’s invariants, like, e.g., the Reshetikhin-Turaev invariant. It is often (but not always) assumed in the literature that the value  $\text{WLO}(L)$  of a link  $L$  in  $M$  associated to the CS path integral with group  $G$ , base manifold  $M$  and level  $k$  is given by the Reshetikhin-Turaev invariant  $\tau_q(L)$  associated to  $G$ ,  $M$  and the root of unity  $q = \exp(\frac{2\pi i}{k+c_\mathfrak{g}})$ . In the special case  $M = \Sigma \times S^1$  the Reshetikhin-Turaev invariant can be expressed by the shadow invariant. In this case the parameter  $q = \exp(\frac{2\pi i}{k+c_\mathfrak{g}})$  of the RT-invariant corresponds to the situation where the parameter  $k$  in Eqs. (B.3)–(B.8) above is replaced by the “shifted” value<sup>83</sup>  $k + c_\mathfrak{g}$ .

Let us emphasize, however, that the necessity/appropriateness of the “shift”  $k \rightarrow k + c_\mathfrak{g}$  is not universally accepted, cf. [20, 24]. In fact, Theorem 5.3 of the present paper supports the view in<sup>84</sup> [24] that the occurrence (and magnitude) of such a shift in  $k$  will depend on the regularization procedure/renormalization prescription which is used.

<sup>81</sup>We remark that, alternatively,  $N_{\mu\nu}^\lambda$  can also be defined with the help of the  $S$ -matrix of the WZW model corresponding to  $\Sigma$ ,  $G$ , and the level  $l := k - c_\mathfrak{g}$ , cf. [47]

<sup>82</sup>We remark that there are different conventions for the definition of  $U_q(\mathfrak{g}_\mathbb{C})$ . Accordingly, one finds different formulas for  $q$  in the literature. For example, using the convention in [47] one would be led to the formula  $q := e^{\frac{\pi i}{Dk}}$  where  $D$  is the quotient of the square lengths of the long and the short roots of  $\mathfrak{g}$

<sup>83</sup>this is the reason why we used a different convention in [18] for the definition of  $\Lambda_+^k$ , namely  $\Lambda_+^k := \{\lambda \in \Lambda_+ \mid (\lambda, \theta) \leq k\}$

<sup>84</sup>see, in particular, p. 599 in Sec. 5 in [24]

## C Appendix: $BF_3$ -theory in the torus gauge

### C.1 $BF_3$ -theory

Let  $M$  be a closed oriented 3-manifold, let  $\tilde{G}$  be a simple simply-connected compact Lie group with Lie algebra  $\tilde{\mathfrak{g}}$ , and let  $\tilde{\mathcal{G}} := C^\infty(M, \tilde{G})$ .

For  $\tilde{A} \in \tilde{\mathcal{A}} := \Omega^1(M, \tilde{\mathfrak{g}})$  and  $\tilde{C} \in \tilde{\mathcal{C}} := \Omega^1(M, \tilde{\mathfrak{g}})$  and  $\Lambda \in \mathbb{R}$  (the “cosmological constant”) we define<sup>85</sup>

$$S_{BF}(\tilde{A}, \tilde{C}) := \frac{1}{\pi} \int_M \text{Tr}(F^{\tilde{A}} \wedge \tilde{C} + \frac{\Lambda}{3} \tilde{C} \wedge \tilde{C} \wedge \tilde{C}) \quad (\text{C.1})$$

where  $F^{\tilde{A}} := d\tilde{A} + \tilde{A} \wedge \tilde{A}$ . Let us assume in the following that  $\Lambda \in \mathbb{R}_+$  and set

$$\kappa := \sqrt{\Lambda} \quad (\text{C.2})$$

Note that  $S_{BF} : \tilde{\mathcal{A}} \times \tilde{\mathcal{C}} \rightarrow \mathbb{C}$  is  $\tilde{\mathcal{G}}$ -invariant under the  $\tilde{\mathcal{G}}$ -operation on  $\tilde{\mathcal{A}} \times \tilde{\mathcal{C}}$  given by  $(A, C) \cdot \tilde{\Omega} = (\tilde{\Omega}^{-1} A \tilde{\Omega} + \tilde{\Omega}^{-1} d\tilde{\Omega}, \tilde{\Omega}^{-1} C \tilde{\Omega})$ .

It is well-known that in the situation  $\kappa \neq 0$  the relation

$$S_{BF}(\tilde{A}, \tilde{C}) = S_{CS}(\tilde{A} + \kappa \tilde{C}) - S_{CS}(\tilde{A} - \kappa \tilde{C}) \quad (\text{C.3})$$

holds with  $S_{CS} = S_{CS}(M, G, k)$  where  $G := \tilde{G}$  and  $k := \frac{1}{\kappa}$ . Using the change of variable  $(\tilde{A}, \tilde{C}) \rightarrow (A_1, A_2)$  given by

$$A_1 := \tilde{A} + \kappa \tilde{C}, \quad A_2 := \tilde{A} - \kappa \tilde{C} \quad (\text{C.4})$$

or, equivalently,

$$\tilde{A} := \frac{1}{2}(A_1 + A_2), \quad \tilde{C} := \frac{1}{2\kappa}(A_1 - A_2) \quad (\text{C.5})$$

we therefore obtain, informally, for every  $\tilde{\chi} : \tilde{\mathcal{A}} \times \tilde{\mathcal{C}} \rightarrow \mathbb{C}$

$$\begin{aligned} & \iint \tilde{\chi}(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A} D\tilde{C} \\ & \sim \iint \chi((A_1, A_2)) \exp(iS_{CS}(A_1)) \exp(-iS_{CS}(A_2)) DA_1 DA_2 \end{aligned} \quad (\text{C.6})$$

where  $\chi : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathbb{C}$  with  $\mathcal{A}_j := \Omega^1(M, \mathfrak{g})$ ,  $j = 1, 2$ , is the function given by  $\chi((A_1, A_2)) = \tilde{\chi}(\tilde{A}, \tilde{C})$ .

If — instead of setting  $S_{CS} := S_{CS}(M, G, k)$  with  $G := \tilde{G}$  and  $k := \frac{1}{\kappa}$  — we use  $S_{CS} := S_{CS}(M, G, (k_1, k_2))$  with  $G = \tilde{G} \times \tilde{G}$  and  $(k_1, k_2) = (1/\kappa, -1/\kappa)$  (cf. Remark 2.7 in [30]) then we can<sup>86</sup> rewrite Eq. (C.6) as

$$\iint_{\tilde{\mathcal{A}} \times \tilde{\mathcal{C}}} \tilde{\chi}(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A} D\tilde{C} \sim \int_{\mathcal{A}} \chi(A) \exp(iS_{CS}(A)) DA \quad (\text{C.7})$$

Thus we see that  $BF_3$ -theory on  $M$  with group  $\tilde{G}$  and  $\kappa \neq 0$  is essentially equivalent to CS theory on  $M$  with group  $G = \tilde{G} \times \tilde{G}$  and  $(k_1, k_2) = (1/\kappa, -1/\kappa)$ .

<sup>85</sup>Here we assume for simplicity (cf. Remark 2.1 above) that  $\tilde{G}$  is Lie subgroup of  $U(\tilde{N})$  for some  $\tilde{N} \in \mathbb{N}$  and we set  $\text{Tr} := \tilde{c} \text{Tr}_{\text{Mat}(\tilde{N}, \mathbb{C})}$  where  $\tilde{c} \in \mathbb{R}$  is chosen suitably

<sup>86</sup>using  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$  and  $DA = DA_1 DA_2$

## C.2 $BF_3$ -theory on $M = \Sigma \times S^1$ “in the torus gauge”

Let us now consider the special case where  $M = \Sigma \times S^1$ ,  $\kappa \neq 0$  and  $1/\kappa \in \mathbb{N}$ , and where  $\tilde{\chi} : \tilde{\mathcal{A}} \times \tilde{\mathcal{C}} \rightarrow \mathbb{C}$  is of the form

$$\tilde{\chi}(\tilde{A}, \tilde{C}) = \prod_{i=1}^m \text{Tr}_{\rho_i}(\text{Hol}_{l_i}(\tilde{A} + \kappa \tilde{C}, \tilde{A} - \kappa \tilde{C})) \quad (\text{C.8})$$

(with  $(l_1, l_2, \dots, l_m)$  and  $(\rho_1, \rho_2, \dots, \rho_m)$  as in Sec. 2.1).

Let us now apply “torus gauge fixing”<sup>87</sup> to the expression

$$\iint_{\tilde{\mathcal{A}} \times \tilde{\mathcal{C}}} \tilde{\chi}(\tilde{A}, \tilde{C}) \exp(iS_{BF}(\tilde{A}, \tilde{C})) D\tilde{A} D\tilde{C} \quad (\text{C.9})$$

More precisely, we will perform the following three steps:

- we make a change of variable from “BF-variables” to “CS-variables”<sup>88</sup> (Step 1)
- we apply torus gauge fixing (Step 2)
- we change back to “BF-variables” (Step 3)

Concretely, these three steps are given as follows:

**Step 1:** We replace the expression (C.9) by the RHS of Eq. (C.7)

**Step 2:** We perform torus gauge-fixing on the RHS of Eq. (C.7), i.e. we replace the RHS of Eq. (C.7) by the RHS of<sup>89</sup> Eq. (2.8) in Sec. 2.2 above (in the situation  $G = \tilde{G} \times \tilde{G}$ ,  $T := \tilde{T} \times \tilde{T}$ , and  $(k_1, k_2) = (1/\kappa, -1/\kappa)$  where  $\tilde{T}$  is a fixed maximal torus of  $\tilde{G}$ )

**Step 3:** We apply the change of variable  $(\check{A}^\perp, A_c, B) \rightarrow (\check{A}^\perp, \tilde{A}_c, \tilde{B})$  given by

$$\check{A}^\perp := \left( \frac{\check{A}_1^\perp + \check{A}_2^\perp}{2}, \frac{\check{A}_1^\perp - \check{A}_2^\perp}{2\kappa} \right), \quad (\text{C.10a})$$

$$\tilde{A}_c := \left( \frac{(A_c)_1 + (A_c)_2}{2}, \frac{(A_c)_1 - (A_c)_2}{2\kappa} \right), \quad (\text{C.10b})$$

$$\tilde{B} := \left( \frac{B_1 + B_2}{2}, \frac{B_1 - B_2}{2\kappa} \right) \quad (\text{C.10c})$$

to the RHS of Eq. (2.8) (in the situation  $G = \tilde{G} \times \tilde{G}$ ,  $T := \tilde{T} \times \tilde{T}$ , and  $(k_1, k_2) = (1/\kappa, -1/\kappa)$ )

The expression which we obtain after performing the three steps above is the analogue of the RHS of Eq. (7.3) above where instead of the change of variable (7.2) the change of variable (C.10) is used.

## D Elimination of point (D2) of Approach II, cf. Sec. 3.4

### D.1 First method

We modify Approach II in the following way:

We replace the space  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  by the space  $\mathcal{A}_{\Sigma, \mathfrak{t}}(K)$  and the scalar product  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})}$  by the scalar product  $\ll \cdot, \cdot \gg_{\mathcal{A}_{\Sigma, \mathfrak{t}}(K)}$ . Moreover, we replace the space  $\mathcal{E}(b\mathcal{K}) = C^2(b\mathcal{K}, \mathfrak{t})$  by the subspace<sup>90</sup>

$$\mathcal{E}(K) := C^2(K_1, \mathfrak{t}) + C^2(K_2, \mathfrak{t}) \subset \mathcal{E}(b\mathcal{K}) \quad (\text{D.1})$$

<sup>87</sup>observe that the field  $\tilde{C}$  does not transform like a 1-form, so the use of the notion “gauge” in the present context is somewhat misleading

<sup>88</sup>i.e. from  $(\tilde{A}, \tilde{C})$  to  $(A_1, A_2)$

<sup>89</sup>cf. Remark 2.7 in [30]

<sup>90</sup>observe that the sum in Eq. (D.1) is not direct, cf. Convention 1 in Sec. 3.1 above



and we replace the scalar product  $\ll \cdot, \cdot \gg_{\mathcal{E}(b\mathcal{K})}$  and the two maps<sup>91</sup>  $\partial_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  and<sup>92</sup>  $\star_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{B}(b\mathcal{K})$  by their restrictions onto  $\mathcal{E}(K)$ . Here we have identified the space  $\mathcal{A}_{\Sigma, \mathfrak{t}}(K) = C^1(K_1, \mathfrak{t}) \oplus C^1(K_2, \mathfrak{t}) \cong C_1(K) \otimes_{\mathbb{R}} \mathfrak{t}$  with a suitable subspace of  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K}) \cong C_1(b\mathcal{K}) \otimes_{\mathbb{R}} \mathfrak{t}$ .

The other constructions/definitions of Approach II remain unchanged.

## D.2 Second method

If we use the polyhedral cell complex  $q\mathcal{K}$  introduced in Remark 3.1 above instead of  $b\mathcal{K}$  then there is an alternative way for eliminating (D2). Let us introduce the spaces

$$\begin{aligned}\mathcal{A}^\perp(q\mathcal{K}) &:= \text{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(q\mathcal{K})) = \text{Map}(\mathbb{Z}_N, C^1(q\mathcal{K}, \mathfrak{g})) \\ \mathcal{A}_\pm^\perp(q\mathcal{K}) &:= \{A^\perp \in \mathcal{A}^\perp(q\mathcal{K}) \mid \forall t \in \mathbb{Z}_N : A^\perp(t)(e_1) = \pm A^\perp(t)(e_2) \text{ if } e_1 \text{ and } e_2 \text{ are conjugated}\} \\ \check{\mathcal{A}}^\perp(q\mathcal{K}) &:= \{A^\perp \in \mathcal{A}^\perp(q\mathcal{K}) \mid \sum_{t \in \mathbb{Z}_N} A^\perp(t) \in \mathcal{A}_{\Sigma, \mathfrak{t}}(q\mathcal{K})\} \\ \mathcal{A}_c^\perp(q\mathcal{K}) &:= \{A^\perp \in \mathcal{A}^\perp(q\mathcal{K}) \mid A^\perp(\cdot) \text{ is constant and } \mathcal{A}_{\Sigma, \mathfrak{t}}(q\mathcal{K})\text{-valued}\}\end{aligned}$$

Here we call two edges  $e_1, e_2 \in \mathcal{F}_1(q\mathcal{K})$  “conjugated” iff their union  $e_1 \cup e_2$  is an edge in  $K_1$  or  $K_2$ . It is not difficult to see that

$$\begin{aligned}\mathcal{A}^\perp(q\mathcal{K}) &= \mathcal{A}_+^\perp(q\mathcal{K}) \oplus \mathcal{A}_-^\perp(q\mathcal{K}) \\ \mathcal{A}_\pm^\perp(q\mathcal{K}) &\cong \mathcal{A}^\perp(K) \\ \mathcal{A}_c^\perp(q\mathcal{K}) &\cong \mathcal{A}_{\Sigma, \mathfrak{t}}(q\mathcal{K}) = C^1(q\mathcal{K}, \mathfrak{t})\end{aligned}$$

Moreover, the linear isomorphism  $\star_K : \mathcal{A}^\perp(K) \rightarrow \mathcal{A}^\perp(K)$  can be extended to a linear isomorphism<sup>93</sup>  $\star'_K : \mathcal{A}^\perp(q\mathcal{K}) \rightarrow \mathcal{A}^\perp(q\mathcal{K})$  in a natural way.

In Secs 4.1–4.8 we now replace the space  $\check{\mathcal{A}}^\perp(K)$  by  $\check{\mathcal{A}}^\perp(q\mathcal{K})$  and the operator  $\star_K$  by  $\star'_K$ . Moreover, we replace  $\mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  by  $\mathcal{A}_{\Sigma, \mathfrak{t}}(q\mathcal{K}) \cong \mathcal{A}_c^\perp(q\mathcal{K})$  and all other constructions depending on  $b\mathcal{K}$  by their “ $q\mathcal{K}$ -analogues”.

In order to verify that a version of Theorem 5.3 will then again hold it is enough to observe that the integration  $\int_{\sim} \cdots D\check{\mathcal{A}}^\perp$  which appears in the new version of Eq. (4.17) can be written as a double integral  $\int_{\sim} \int_{\sim} \cdots D\check{\mathcal{A}}_-^\perp D\check{\mathcal{A}}_+^\perp$ . (Here  $D\check{\mathcal{A}}^\perp$  is the Lebesgue measure on  $\check{\mathcal{A}}^\perp(q\mathcal{K})$  and  $D\check{\mathcal{A}}_\pm^\perp$  is the Lebesgue measure on  $\check{\mathcal{A}}_\pm^\perp(q\mathcal{K}) := \check{\mathcal{A}}^\perp(q\mathcal{K}) \cap \mathcal{A}_\pm^\perp(q\mathcal{K})$ ). It turns out that the integrand depends on  $\check{\mathcal{A}}^\perp$  in a trivial way. Thus the  $\int_{\sim} \cdots D\check{\mathcal{A}}^\perp$ -integration is trivial and produces just a constant factor. Since  $\check{\mathcal{A}}_+^\perp(q\mathcal{K}) \cong \check{\mathcal{A}}^\perp(K)$  it is clear that the formula at which we arrive after performing the  $\int_{\sim} \cdots D\check{\mathcal{A}}^\perp$ -integration is the “ $q\mathcal{K}$ -analogue” of Eq. (4.17) and can be evaluated explicitly in exactly the same way as the original Eq. (4.17).

## E Two alternative ways of discretizing the RHS of Eq. (7.7)

### E.1 Alternative 1

In part D of the Appendix of the present paper and Appendix D in [30] we mentioned some ideas which should lead to a stylistic improvement of Approach I and Approach II. In fact, we can apply the aforementioned ideas (in a slightly modified form) also to the situation in Sec. 7.3. In the following we will use the idea of Appendix D in [30] and the first method mentioned in part D of the Appendix of the present paper:

<sup>91</sup>observe that  $\partial_{b\mathcal{K}}(\mathcal{E}(K)) \subset \mathcal{A}_{\Sigma, \mathfrak{t}}(K)$  so the restriction of  $\partial_{b\mathcal{K}} : \mathcal{E}(b\mathcal{K}) \rightarrow \mathcal{A}_{\Sigma, \mathfrak{t}}(b\mathcal{K})$  onto  $\mathcal{E}(K)$  will be a well-defined map  $\mathcal{E}(K) \rightarrow \mathcal{A}_{\Sigma, \mathfrak{t}}(K)$

<sup>92</sup>we make similar replacements for the other three of the four maps given in Convention 4 in Sec. 4.1

<sup>93</sup>we remark that  $\star'_K$  does not fulfill the relation  $(\star'_K)^2 = -\text{id}$  but only the weaker condition  $(\star'_K)^4 = \text{id}$  and is therefore slightly less natural than the operator  $\star_K$

Instead of using the spaces  $\mathcal{E}(b\mathcal{K}) = C^2(b\mathcal{K}, \mathfrak{t} \oplus \mathfrak{t})$  and  $\mathcal{A}_c(b\mathcal{K}) := C^1(b\mathcal{K}, \mathfrak{t} \oplus \mathfrak{t})$  as in Sec. 7.3 let us now use the two spaces

$$\begin{aligned}\mathcal{E}(K) &:= C^2(K_1, \mathfrak{t} \oplus \mathfrak{t}) + C^2(K_2, \mathfrak{t} \oplus \mathfrak{t}) \subset \mathcal{E}(b\mathcal{K}), \\ \mathcal{A}_c(K) &:= C^1(K_1, \mathfrak{t} \oplus \mathfrak{t}) \oplus C^1(K_2, \mathfrak{t} \oplus \mathfrak{t})\end{aligned}$$

We set

$$\mathcal{A}^\perp(K) := \text{Map}(\mathbb{Z}_{2N}, \mathcal{A}_\Sigma(K)) = \text{Map}(\mathbb{Z}_{2N}, C^1(K_1, \mathfrak{g}) \oplus C^1(K_2, \mathfrak{g}))$$

and equip this space with the “obvious” scalar product, i.e. the one given by

$$\ll A_1^\perp, A_2^\perp \gg_{\mathcal{A}^\perp(K)} = \frac{1}{2N} \sum_{t \in \mathbb{Z}_{2N}} \ll A_1^\perp(t), A_2^\perp(t) \gg_{\mathcal{A}_\Sigma(K)} \quad (\text{E.1})$$

where  $\ll \cdot, \cdot \gg_{\mathcal{A}_\Sigma(K)}$  is as in the paragraph before Eq. (2.16) in Sec. 2.3 above. (We remark that in contrast to the scalar product given in Eq. (2.16) above the summation is now over  $\mathbb{Z}_{2N}$  and not only over  $\mathbb{Z}_N$ ).

For  $A^\perp \in \mathcal{A}^\perp(K) = \text{Map}(\mathbb{Z}_{2N}, \mathcal{A}_\Sigma(K))$  we will denote by  $A_1^\perp$  and  $A_2^\perp$  the components of  $A^\perp$  w.r.t. the following decomposition

$$\mathcal{A}^\perp(K) = \mathcal{A}_{\text{altern},1}^\perp(K) \oplus \mathcal{A}_{\text{altern},2}^\perp(K) \quad (\text{E.2})$$

where

$$\begin{aligned}\mathcal{A}_{\text{altern},1}^\perp(K) &:= \text{Map}(\mathbb{Z}_{2N}^{\text{odd}}, C^1(K_1, \mathfrak{g})) \oplus \text{Map}(\mathbb{Z}_{2N}^{\text{even}}, C^1(K_2, \mathfrak{g})) \\ \mathcal{A}_{\text{altern},2}^\perp(K) &:= \text{Map}(\mathbb{Z}_{2N}^{\text{odd}}, C^1(K_2, \mathfrak{g})) \oplus \text{Map}(\mathbb{Z}_{2N}^{\text{even}}, C^1(K_1, \mathfrak{g}))\end{aligned}$$

and with  $\mathbb{Z}_{2N}^{\text{odd}}$  and  $\mathbb{Z}_{2N}^{\text{even}}$  as in Appendix D in [30]. (Observe that the space  $\mathcal{A}_{\text{altern},1}^\perp(K)$  above coincides with the space called  $\mathcal{A}_{\text{altern}}^\perp(K)$  in Appendix D in [30]).

There is a second direct sum decomposition  $\mathcal{A}^\perp(K) = \check{\mathcal{A}}^\perp(K) \oplus \mathcal{A}_c^\perp(K)$  which is analogous to the decomposition (2.14) in Sec. 2.3 above. So every  $A^\perp \in \mathcal{A}^\perp(K)$  can be written as a sum of four terms<sup>94</sup>  $A^\perp = \check{A}_1^\perp + (A_c)_1 + \check{A}_2^\perp + (A_c)_2$ . Using this decomposition we will now reinterpret Eqs. (7.12)–(7.14) in Sec. 7.3 above in the obvious way.

Finally, recall that the loops  $l_i$  and the framings  $l'_i$ ,  $i \leq m$  fixed in Sec. 4.8 above are (discrete) loops in  $K_1 \times \mathbb{Z}_N$  and  $K_2 \times \mathbb{Z}_N$ , respectively. By identifying the set  $\mathfrak{F}_0(K_j \times \mathbb{Z}_N)$ ,  $j = 1, 2$  with the “obvious” subset of<sup>95</sup>  $\mathfrak{F}_0(K_j \times \mathbb{Z}_{2N}^{\text{even}})$  we can consider these loops as loops in  $K_j \times \mathbb{Z}_{2N}^{\text{even}}$ ,  $j = 1, 2$ .

The remaining steps for discretizing the RHS of Eq. (7.7) can be carried out in a straightforward way. We expect that by doing so one arrives at a variant of Theorem 7.4 which will/would have the following (stylistic) advantages over the original version of Theorem 7.4:

- Point (D2) in Sec. 3.4 above is eliminated.
- The modified version of the operator  $R^{(N)}(B)$  appearing in Sec. 7.3 will be more natural (cf. Remark 3.10 above).
- The scalar product in Eq. (E.1) is more natural than the scalar products  $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp(K)}$  and  $\ll \cdot, \cdot \gg_{\check{\mathcal{A}}^\perp(K)}$  used in the main text of this paper, cf. Eq. (2.16) above.

<sup>94</sup>  $\check{A}_j^\perp$  lies in the intersection  $\mathcal{A}_{\text{altern},j}^\perp(K) \cap \check{\mathcal{A}}^\perp(K)$  and  $(A_c)_j$  lies in the intersection  $\mathcal{A}_{\text{altern},j}^\perp(K) \cap \mathcal{A}_c^\perp(K)$

<sup>95</sup> here we have equipped  $\mathbb{Z}_{2N}^{\text{even}}$  with the graph structure which is analogous to the graph structure of  $\mathbb{Z}_N$

## E.2 Alternative 2

The two approaches for discretization the RHS of Eq. (7.7) which we sketched in Sec. 7.3 and in E.1 above are both in the spirit of Approach II of the first half of the present paper. For the sake of completeness let us sketch briefly an alternative way of discretizing Eq. (7.7) (or, equivalently, Eq. (7.3) above) which is closer to Approach I in [30] (and also closer to the Approach in Adams for Abelian groups) and which should lead to another version of Theorem 7.4 above:

Instead of using the heuristic formula Eq. (7.7) above as the starting point for a discretization we begin with the original formula (7.3) above, or rather, a suitably rewritten version of (7.3) where we again drop all the  $\sim$ -signs in a similar way as in Sec. 7.3.

As the discrete analogues of the space  $\check{\mathcal{A}}^\perp$  we use again the space  $\check{\mathcal{A}}^\perp(K)$  as in Sec. 7.3 and as the discrete analogues of the spaces  $\mathcal{B}$ ,  $\mathcal{A}_c$  we now use the spaces

$$\begin{aligned}\mathcal{B}(K) &:= C^0(K_1, \mathfrak{t}) \oplus C^0(K_2, \mathfrak{t}), \\ \mathcal{A}_c(K) &:= C^1(K_1, \mathfrak{t}) \oplus C^1(K_2, \mathfrak{t})\end{aligned}$$

Clearly, within this setting we do not need an operator of the type  $\star_{bK}$  but can work with the operator  $\star_K$  in Eq. (2.18) in Remark 2.4.

However, in order to define the discrete analogue of  $B_\pm = B_1 \pm B_2$  we again have to solve an extension problem, one that is similar to the extension problem related to the operator  $\star_{bK}$ , cf. point (D1) in Sec. 3.4. Moreover, observe that within the ansatz just described the two spaces  $\check{\mathcal{A}}^\perp(K)$  and  $\mathcal{A}_c(K)$  clearly<sup>96</sup> “do not combine well” in the same sense as in point (D2) in Sec. 3.4.

Summarizing this we can say that the closest thing which one gets when one tries to find a BF-version of Approach I in [30] leading to a result like Theorem 7.4 with realistic chances<sup>97</sup> of holding also for general links shows similar features as points (D1) and (D2) in Approach II.

## F Some remarks on the simplicial program for CS-theory/ $BF_3$ -theory

The goal of what is called the “simplicial program” for CS-theory/ $BF_3$ -theory in [41] (cf. also [5]) is to find a discretized and rigorous version of the *non-gauged fixed* CS or  $BF_3$  path integral for the WLOs associated to links in arbitrary oriented closed 3-manifolds  $M$  such that the values of these discretized path integrals coincide with the values of the corresponding<sup>98</sup> Reshetikhin-Turaev-Witten invariants. This discretization is supposed to involve only finite triangulations (or, more generally, finite polyhedral cell decompositions) of  $M$  and no continuum limit.

As mentioned in Sec. 7.1 above the simplicial program for  $CS/BF_3$ -theory was completed successfully for Abelian structure groups in [1, 2] (cf. also [33] for a short review of Adams’ work). The case of non-Abelian structure groups is an important open problem.

One possible strategy for making progress in the simplicial program for non-Abelian  $CS/BF_3$ -theory could be to try to complete, successively, the following three “projects”:

**Project 1** Find a simplicial definition of the WLOs associated to general links for non-Abelian  $CS$ -theory/ $BF_3$ -theory on the special base manifold  $M = \Sigma \times S^1$  in the torus gauge.

<sup>96</sup>since we have set  $\mathcal{A}_c(K) := C^1(K_1, \mathfrak{t}) \oplus C^1(K_2, \mathfrak{t})$  and not  $\mathcal{A}_c(K) := C^1(K_1, \mathfrak{t} \oplus \mathfrak{t}) \oplus C^1(K_2, \mathfrak{t} \oplus \mathfrak{t})$

<sup>97</sup>We remark that a “complete”  $BF$ -analogue of Approach I (including the features (Mod2) and (Mod3) in Sec. 5.10 of [30]) would not have “realistic chances” since in such an approach we would again have the problem of extra  $1/2$ -factors if we deal with general links, cf. the last paragraph of Sec. 6 above

<sup>98</sup>more precisely: if  $G$  and  $k$  are as in Sec. 2.1,  $M$  is a general oriented closed 3-manifold,  $L$  a link in  $M$ , and  $\tau_q(M, L)$ ,  $q = \exp(2\pi i/k)$  the corresponding Reshetikhin-Turaev-Witten invariant then we want that the discretized rigorous version of  $WLO(L)$  coincides with  $\tau_q(M, L)$  up to a suitable multiplicative constant

Clearly, Project 1 is exactly what we are dealing with in [30], the present paper, and its sequel [31]. Theorem 7.4 above can be seen as a first step towards the completion of Project 1. In order to complete Project 1 successfully Theorem 7.4 will have to be generalized to arbitrary links in  $M = \Sigma \times S^1$ .

**Project 2** Find a simplicial definition of the WLOs associated to general links for the *non-gauge fixed* non-Abelian  $CS$ -theory/ $BF_3$ -theory on  $M = \Sigma \times S^1$ .

Observe that there is a natural “discrete” analogue of the torus gauge fixing procedure. So if one can complete Project 1 successfully there might be a quick way to complete also Project 2. In order to do so we could look at the simplicial definition of the WLOs used in Project 1 and then try to “reverse engineer” from it a non-gauge fixed “version”, i.e. a suitable simplicial expression which – after applying “discrete” torus gauge fixing – leads to the simplicial expression used in Project 1.

**Project 3** Generalize the simplicial expressions for the WLOs at which we arrive in Project 2 to arbitrary  $M$  and evaluate these expressions explicitly.

One can speculate that if Project 2 could be carried out successfully then the chances for a successful completion of Project 3 would be quite good. All we would have to do then is to find a rigorous implementation of Witten’s surgery procedure, see [32] for some ideas in this direction.

At the moment it is completely open whether these three “Projects” can indeed be carried out successfully. For Project 2 it seems to be necessary to find a way to bypass/eliminate the following “complications”:

**Complication (C1)** Recall the definition of  $\text{Det}_{FP}^{disc}(B)$  in Eq. (4.4). We had

$$\text{Det}_{FP}^{disc}(B) := \prod_{x \in \mathfrak{F}_0(b\mathcal{K})} \det(1_{\mathfrak{k}} - \exp(\text{ad}(B(x))))_{|\mathfrak{k}}^{1/2}$$

Discrete torus gauge fixing (cf. the second paragraph in Project 2 above) is related to the map  $q : G/T \times T \rightarrow G$  given by  $q(\bar{g}, t) = \text{Ad}(g)t$  for  $t \in T$  and  $\bar{g} = gT \in G/T$ . Since  $(\bar{g}, t) \mapsto \det(1_{\mathfrak{k}} - \text{Ad}(t))_{|\mathfrak{k}}$  is the “Jacobian”<sup>99</sup> of the map  $q$  it is easy to see how factors of the form  $\det(1_{\mathfrak{k}} - \exp(\text{ad}(B(x))))_{|\mathfrak{k}} = \det(1_{\mathfrak{k}} - \text{Ad}(\exp(B(x))))_{|\mathfrak{k}}$  in Eq. (4.4) above can arise from discrete torus gauge fixing.

However, it is not clear how the  $1/2$ -exponents in the expression Eq. (4.4) and in Eq. (5.34) in [30] can arise from such a procedure. Moreover, for each of these two equations there is an additional problem:

In Eq. (5.34) in [30] it is not clear how/why the factors  $\det(1_{\mathfrak{k}} - \text{Ad}(B(\bar{e})))_{|\mathfrak{k}}$ ,  $\bar{e} \in \mathfrak{F}_0(K_1|K_2)$ , should arise since the expressions  $B(\bar{e})$  are not independent variables but functions of the variables  $\{B_1(x) \mid x \in \mathfrak{F}_0(K_1)\}$  and  $\{B_2(x) \mid x \in \mathfrak{F}_0(K_2)\}$ , cf. Eq. (5.11) and/or (5.35) in [30].

In Eq. (4.4) all the vertices  $x \in \mathfrak{F}_0(b\mathcal{K})$  are on equal footing but the field  $B$  appearing in Eq. (4.4) is a function of the field  $E$  which appears as the integration variable in Eq. (4.17). So in the setting of Approach II it is not clear how/why a product of the form  $\prod_{x \in \mathfrak{F}_0(b\mathcal{K})} \cdots$  rather than one of the form  $\prod_{F \in \mathfrak{F}_2(b\mathcal{K})} \cdots$  should arise from discrete torus gauge fixing.

<sup>99</sup>more precisely, we have  $q^*(\nu_G) = \det(1_{\mathfrak{k}} - \text{Ad}(\pi_2(\cdot)))_{|\mathfrak{k}} (\pi_1^*(\nu_{G/T}) \wedge \pi_2^*(\nu_T))$  where  $\nu_G$ ,  $\nu_T$ , and  $\nu_{G/T}$  are the canonical volume forms on  $G$ ,  $T$ , and  $G/T$  and  $\pi_1 : G/T \times T \rightarrow G/T$  and  $\pi_2 : G/T \times T \rightarrow T$  the canonical projections

**Complication (C2)** The traces of the holonomies associated to “single” (=unframed) loops are gauge-invariant functions. However, in Approach I, in Approach II, and in the approach sketched in Sec. 7 we used framed loops for reasons explained in Remark 5.5 above. In contrast to the traces of the holonomies associated to single loops the traces of the holonomies associated to framed loops are no-longer gauge-invariant functions. In other words: there is no natural candidate for a “non-gauge fixed version” of the expressions<sup>100</sup>  $\text{Tr}_{\rho_i}(\text{Hol}_{l_i}^{disc}(\check{A}^\perp, A_c, B; h))$  and  $\text{Tr}_{\rho_i}(\text{Hol}_{l_i}^{disc}(A_1^\perp + A_2^\perp, B_+, h_+))$ , cf. Eq. (4.3) and Eq. (7.14) above.

**Complication (C3)** For discrete torus gauge fixing there are no topological obstructions like in the case of (continuum) torus gauge fixing. Accordingly, it is not clear what the discrete analogue of the 1-forms  $A_{\text{sg}}(h)$  appearing in Eq. (3.6) above could be.

If one can justify the simplified versions of Eqs. (2.8) and (3.6) mentioned in Remark 2.2 and Remark 3.2 above the situation improves since the 1-forms  $A_{\text{sg}}(h)$  then do not appear anymore, cf. Eq. (2.12) above. The remaining problem will be to explain how a summation of the type  $\sum_{x \in I}$  and the factors  $\exp(-2\pi i k \langle x, B(\sigma_0) \rangle)$  in Eq. (2.12) can arise from the simplicial version of the non-gauge fixed CS path integral after applying discrete torus gauge fixing.

**Remark F.1** Let us remark that point (D2) in the original version of Approach II (cf. Sec. 3.4) would be another important obstacle for Project 2. This is the main reason why in part D of the Appendix above we sketched two different modifications of Approach II which allows us to eliminate point (D2).

## References

- [1] D. H. Adams. R-torsion and linking numbers from simplicial abelian gauge theories. Preprint, arXiv:hep-th/9612009, 1996.
- [2] D. H. Adams. A doubled discretization of abelian Chern-Simons theory. *Phys. Rev. Lett.*, 78(22):4155–4158, 1997.
- [3] S. Albeverio and A.N. Sengupta. A Mathematical Construction of the Non-Abelian Chern-Simons Functional Integral. *Commun. Math. Phys.*, 186:563–579, 1997.
- [4] J.E. Andersen. The asymptotic expansion conjecture, Sec. 7.2. in *Problems on invariants of knots and 3-manifolds*, ed. T. Ohtsuki, in *Invariants of knots and 3-manifolds*, Geometry and Topology Monographs 4 (2002), 377–572.
- [5] M. Atiyah. *The Geometry and Physics of Knots*, Cambridge U. Press, Cambridge, 1990.
- [6] John C. Baez. Four-dimensional  $BF$  theory as a topological quantum field theory. *Lett. Math. Phys.*, 38(2):129–143, 1996.
- [7] J. Barrett and I. Naish-Guzman. The Ponzano-Regge model, arXiv:0803.3319
- [8] C. Beasley and E. Witten. Non-abelian localization for Chern-Simons theory. *J. Differential Geom.*, 70(2):183–323, 2005.
- [9] C. Beasley. Localization for Wilson Loops in Chern-Simons Theory. arXiv:0911.2687, 2009
- [10] M. Blau and G. Thompson. A new class of topological field theories and the Ray-Singer torsion. *Phys. Lett. B*, 228(1):64–68, 1989.

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<sup>100</sup>according to our comments in point (C3) below we can probably drop the 1-forms  $A_{\text{sg}}(h)$  and  $A_{\text{sg}}(h_+)$  appearing on the RHS of Eq. (4.3) and Eq. (7.14)

- [11] M. Blau and G. Thompson. Derivation of the Verlinde Formula from Chern-Simons Theory and the G/G model. *Nucl. Phys.*, B408(1):345–390, 1993.
- [12] M. Blau and G. Thompson. Lectures on 2d Gauge Theories: Topological Aspects and Path Integral Techniques. In E. Gava et al., editor, *Proceedings of the 1993 Trieste Summer School on High Energy Physics and Cosmology*, pages 175–244. World Scientific, Singapore, 1994.
- [13] M. Blau and G. Thompson. On Diagonalization in  $Map(M, G)$ . *Commun. Math. Phys.*, 171:639–660, 1995.
- [14] M. Blau and G. Thompson. Chern-Simons theory on  $S^1$ -bundles: abelianisation and  $q$ -deformed Yang-Mills theory. *J. High Energy Phys.*, (5):003, 35 pp. (electronic), 2006.
- [15] Th. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.
- [16] A. Cattaneo, P. Cotta-Ramusino, J. Fröhlich, and M. Martellini. Topological  $BF$  theories in 3 and 4 dimensions. *J. Math. Phys.*, 36(11):6137–6160, 1995.
- [17] J. Cheeger. Analytic torsion and the heat equation. *Ann. of Math. (2)*, 109(2):259–322, 1979.
- [18] S. de Haro and A. Hahn. Chern-Simons theory and the quantum Racah formula. Preprint, arXiv:math-ph/0611084.
- [19] D. S. Fine. On Chern-Simons and WZW partition functions. *Comm. Math. Phys.*, 200(3):685–698, 1999.
- [20] L. Freidel. Private communication, 2008.
- [21] L. Freidel and D. Louapre. Ponzano-Regge model revisited. I. Gauge fixing, observables and interacting spinning particles. *Classical Quantum Gravity*, 21(24):5685–5726, 2004.
- [22] L. Freidel and D. Louapre. Ponzano-Regge model revisited II: Equivalence with Chern-Simons. Preprint, arXiv:gr-qc/0410141v3, 2004.
- [23] L. Freidel and E. R. Livine. Ponzano-Regge model revisited. III. Feynman diagrams and effective field theory. *Classical Quantum Gravity*, 23(6):2021–2061, 2006.
- [24] E. Guadagnini, M. Martellini, M. Mintchev. Braids and quantum group symmetry in Chern-Simons theory. *Nucl. Phys. B*, 336:581–609 (1990).
- [25] A. Hahn. The Wilson loop observables of Chern-Simons theory on  $\mathbb{R}^3$  in axial gauge. *Commun. Math. Phys.*, 248(3):467–499, 2004.
- [26] A. Hahn. Chern-Simons models on  $S^2 \times S^1$ , torus gauge fixing, and link invariants I. *J. Geom. Phys.*, 53(3):275–314, 2005.
- [27] A. Hahn. Chern-Simons models on  $S^2 \times S^1$ , torus gauge fixing, and link invariants II. *J. Geom. Phys.*, 58:1124–1136, 2008.
- [28] A. Hahn. An analytic Approach to Turaev’s Shadow Invariant. *J. Knot Th. Ram.*, 17(11):1327–1385, 2008 [see arXiv:math-ph/0507040v7 (2011) for the most recent version]
- [29] A. Hahn. White noise analysis in the theory of three-manifold quantum invariants. In A.N. Sengupta and P. Sundar, editors, *Infinite Dimensional Stochastic Analysis*, volume XXII of *Quantum Probability and White Noise Analysis*, pages 201–225. World Scientific, 2008.



- [30] A. Hahn. From simplicial Chern-Simons theory to the shadow invariant I, Preprint, 2012.
- [31] A. Hahn. From simplicial Chern-Simons theory to the shadow invariant III, in preparation
- [32] A. Hahn. Surgery operations on the  $BF_3$  path integral via conditional expectations, in preparation
- [33] A. Hahn, B. Himpel, and B. McLellan. Abelian Chern-Simons theory, in preparation.
- [34] T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit. *White Noise. An infinite dimensional Calculus*. Dordrecht: Kluwer, 1993.
- [35] S.-T. Hu. *Homotopy Theory*. Academic Press, New York and London, 1959.
- [36] Y. Katznelson. *An introduction to harmonic analysis, Harmonic Analysis*, Cambridge University Press, 2004
- [37] A. N. Kirillov and N.Y. Reshetikhin. Representations of the algebra  $U_q(sl_2)$ ,  $q$ -orthogonal polynomials and invariants of links. In V.G. Kac et al., editor, *Infinite Dimensional Lie Algebras and Groups*, Vol. 7 of *Advanced Ser. in Math. Phys.*, pages 285–339, 1988.
- [38] D. Krotov and A. Losev. Quantum Field Theory as Effective BV Theory from Chern-Simons. arXiv:hep-th/0603201.
- [39] G. Kuperberg. Quantum invariants of knots and 3-manifolds (book review). *Bull. Amer. Math. Soc.*, 33(1):107–110, 1996.
- [40] J. M. F. Labastida. Chern-Simons Gauge Theory: ten years after, arXiv: hep-th/9905057v1
- [41] P. Mněv. Notes on simplicial BF theory, arXiv: hep-th/0610236v3
- [42] W. Müller. Analytic torsion and  $R$ -torsion of Riemannian manifolds. *Adv. in Math.*, 28(3):233–305, 1978.
- [43] N.Y. Reshetikhin and V.G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Commun. Math. Phys.*, 127:1–26, 1990.
- [44] N.Y. Reshetikhin and V.G. Turaev. Invariants of three manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.
- [45] J. Roberts. Skein theory and Turaev-Viro invariants. *Topology*, 34(4):771–787, 1995.
- [46] L. Rozansky. A contribution of the trivial connection to Jones polynomial and Witten’s invariant of 3d manifolds. I, arXiv:hep-th/9401061
- [47] S. F. Sawin. Quantum groups at roots of unity and modularity. *J. Knot Theory Ramifications*, 15(10):1245–1277, 2006.
- [48] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*. de Gruyter, 1994.
- [49] V.G. Turaev. Shadow links and face models of statistical mechanics. *J. Diff. Geom.*, 36:35–74, 1992.
- [50] E. Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.